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Rates of escape under iteration of analytic functions

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A thesis submitted to

SCHOOL OF MATHEMATICS AND STATISTICS

THE OPEN UNIVERSITY

for the degree of Doctor of Philosophy

September 19, 2016

Declaration

I confirm that the material contained in this thesis is the result of independent work, except where explicitly stated. None of it has previously been submitted for a degree or other qualification to this or any other university or institution.

Vasiliki Evdoridou

September 19, 2016

Abstract

In this thesis we focus on sets with a uniform rate of escape under the iteration of transcendental entire functions. We study their properties and their structure as well as using them as a tool to prove some interesting topological results.

First, motivated by the work of Rippon and Stallard, we generalise the quite fast escaping set by introducing a family of sets that escape to infinity at a uniform rate associated with the maximum modulus of the function. We examine under which conditions these sets are equal to the fast escaping set, which plays an important role in the iteration of transcendental entire functions. We prove that, in some cases, points which satisfy a relatively weak condition are actually fast escaping. We also show that this is not always the case by constructing an example of a function whose fast escaping set is not equal to these newly introduced sets.

Secondly, we look at the so-called spider's web structure and we give some general results which ensure that the escaping set, or a superset thereof, contains a spider's web. In particular, we prove that for a well-known function that was first studied by Fatou the escaping set has the structure of a spider's web. Using similar techniques, we also generalise this result to a class of functions all of which have one Baker domain.

Finally, motivated by the connection between spiders' webs and the non-escaping endpoints of Cantor bouquet Julia sets that first appeared when we studied Fatou's function, we present some topological results on the non-escaping endpoints of functions in the exponential family. More specifically, we show that the set of non-escaping endpoints together with infinity forms a totally separated set for every function in the exponential family whose singular value belongs to the Fatou set. This result is complementary to one of Alhabib and Rempe-Gillen who showed that, for the same class of functions, infinity is an explosion point for the set of escaping endpoints.

Publications

Much of the content of this thesis has previously been published in the form of papers, as follows:

- (1) The results in Chapter 2 have appeared in the Mathematical Proceedings of the Cambridge Philosophical Society [**33**].
- (2) The results in Chapter 3 have been submitted for publication (arXiv : 1603.01519).
- (3) The results on spiders' webs (Sections 4.1-4.5) have appeared in the Proceedings of the American Mathematical Society [**34**].

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CHAPTER 1

Introduction and background

1.1. An introduction to complex dynamics

This thesis is in the area of complex dynamics and, in particular, it focuses on the iteration of transcendental entire functions. Fatou and Julia established the main ideas of one-dimensional complex dynamics in the early 20th century and Fatou was also the first one to study the iteration of transcendental entire functions, in [35]. A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is transcendental entire if it is holomorphic and is not a polynomial. In other words, transcendental entire functions have an essential singularity at infinity. Examples of transcendental entire functions are $z \mapsto e^z$, $z \mapsto \sin^2 z$, $z \mapsto \cosh 2z$. Throughout this thesis, f denotes a transcendental entire function and f^n denotes the n -th iterate of f ; that is, the composition of f with itself n times. For $z \in \mathbb{C}$, the sequence $(f^n(z))_{n \in \mathbb{N}}$ is called the *orbit* of z under f . The set of points $z \in \mathbb{C}$ for which $(f^n)_{n \in \mathbb{N}}$ forms a normal family in some neighbourhood of z is called the *Fatou set* $F(f)$ and the complement of $F(f)$ is the *Julia set* $J(f)$. We say that a family \mathcal{F} of analytic functions on $G \subset \mathbb{C}$ is *normal* if every sequence of functions in \mathcal{F} contains a subsequence that converges locally uniformly on G to an analytic function or to infinity. According to *Montel's theorem*, every family of analytic functions all of which omit the same two values $a, b \in \mathbb{C}$ is normal. The behaviour of the iterates of f is stable in the Fatou set and chaotic in the Julia set. Moreover, $F(f)$ is an open set and so $J(f)$ is closed. An introduction to the properties of these sets can be found in [11]. We say that a set $S \subset \mathbb{C}$ is *forward invariant* if for all $z \in S$ we have $f(z) \in S$, *backward invariant* if for all $z \in S$ we have $f^{-1}(z) \in S$ and *completely invariant* if it is both forward and backward invariant. Both $F(f)$ and $J(f)$ are completely invariant sets.

Notation. In this thesis we use the following notation.

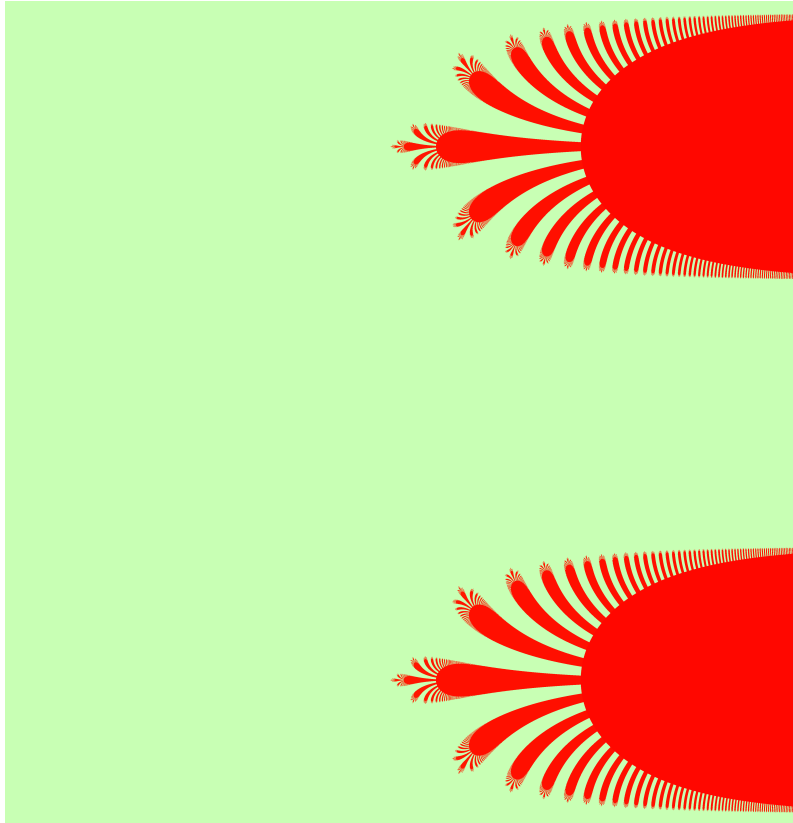
- \mathbb{C} denotes the complex plane
- $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
- $\mathbb{D} = \{z : |z| < 1\}$
- $D(a, r) = \{z : |z - a| < r\}$, $a \in \mathbb{C}$ and $r > 0$
- \tilde{U} denotes the union of U and all its bounded complementary components

1.1.1. Periodic points. A point z_0 is called a *periodic point* of f if there exists $p \in \mathbb{N}$ such that $f^p(z_0) = z_0$. The smallest p with this property is called *the period* of z_0 . If $p = 1$ then z_0 is a fixed point of f . Whenever z_0 is a periodic point of period p the set of points $\{z_0, f(z_0), \dots, f^{p-1}(z_0)\}$ is called a *periodic cycle* of f . For a periodic point z_0 of period p the complex number $(f^p)'(z_0)$ is called the multiplier of z_0 . A periodic point is called *attracting*, *indifferent*, or *repelling* if the modulus of its multiplier is less than, equal to, or greater than 1 respectively. In the case where $(f^p)'(z_0) = 1$, z_0 is called a *parabolic* periodic point.

Periodic points always exist for transcendental entire functions as Fatou showed in [35]. In fact, Rosenbloom proved in [73] that a transcendental entire function has infinitely many periodic points of period p for all $p \geq 2$ and Bergweiler showed later that the same is true for repelling periodic points of transcendental entire functions (see [13]). Attracting periodic points belong to the Fatou set while repelling and parabolic periodic points belong to the Julia set.

1.1.2. The Fatou set. Of great importance in the study of iteration are the connected components of the Fatou set, which are known as Fatou components. A Fatou component U is called *periodic* if there exists $p \in \mathbb{N}$ such that $f^p(U) \subset U$, *preperiodic* if it eventually maps to a periodic component, or a *wandering domain* if $f^n(U) \neq f^m(U)$, for any $n, m \in \mathbb{N}$ with $n \neq m$. Let U_p be the Fatou component which contains $f^p(U)$, $p \in \mathbb{N}$. If U is periodic with period p then $\{U_0 = U, U_1, \dots, U_{p-1}\}$ is called a (periodic) cycle of Fatou components. A periodic Fatou component U is of one of the following four types:

- an *immediate attracting basin* if U contains an attracting periodic point z_0 of period p and $f^{np}(z) \rightarrow z_0$ as $n \rightarrow \infty$ for $z \in U$;
- a *parabolic domain* if ∂U contains a periodic point z_0 of period p and $f^{np}(z) \rightarrow z_0$ as $n \rightarrow \infty$ for $z \in U$. In this case $(f^p)'(z_0) = 1$;

FIGURE 1.1. $J(f)$ for $f_a(z) = e^z + a$

- a *Siegel disc* if there exists an analytic homeomorphism $\phi : U \rightarrow \mathbb{D}$, where \mathbb{D} is the unit disc, such that $\phi(f^p(\phi^{-1}(z))) = e^{2\pi i \alpha} z$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$;
- a *Baker domain* if $f^n(z) \rightarrow \infty$ as $n \rightarrow \infty$ for $z \in U$.

Baker domains and wandering domains do not exist for rational functions. In this thesis we are mostly interested in functions with attracting or parabolic cycles as well as functions with Baker domains.

The most well-studied family of transcendental entire functions is probably the exponential family $f_a(z) = e^z + a$, $a \in \mathbb{C}$, which gives a relatively simple parameter space of transcendental entire functions in analogy to the role that the Mandelbrot set plays in rational dynamics (see [23] and [64]). Several people have studied the exponential family in complex dynamics and proved some of its very interesting dynamical properties; see, for example, [4], [28], [29], [56], [60], [58], [63], [76]. The most detailed information is known

for the case where $a < -1$. In this case f_a has a real fixed point z lying in $(a, 0)$ for which $f'_a(z) = e^z < 1$ and the Fatou set of f_a consists of one component where the iterates converge to the attracting fixed point. So $F(f_a)$ is an immediate attracting basin (see Figure 1.1 ($F(f_a)$ in light green)). As we will see later, when $a \in F(f_a)$ (which includes the case where $a < -1$), f_a has either an attracting or a parabolic cycle of Fatou components. In fact, this is the only case where $F(f_a)$ consists of an attracting or parabolic cycle of Fatou components. Throughout Sections 1.1 and 1.3 we focus on the case where $a < -1$.

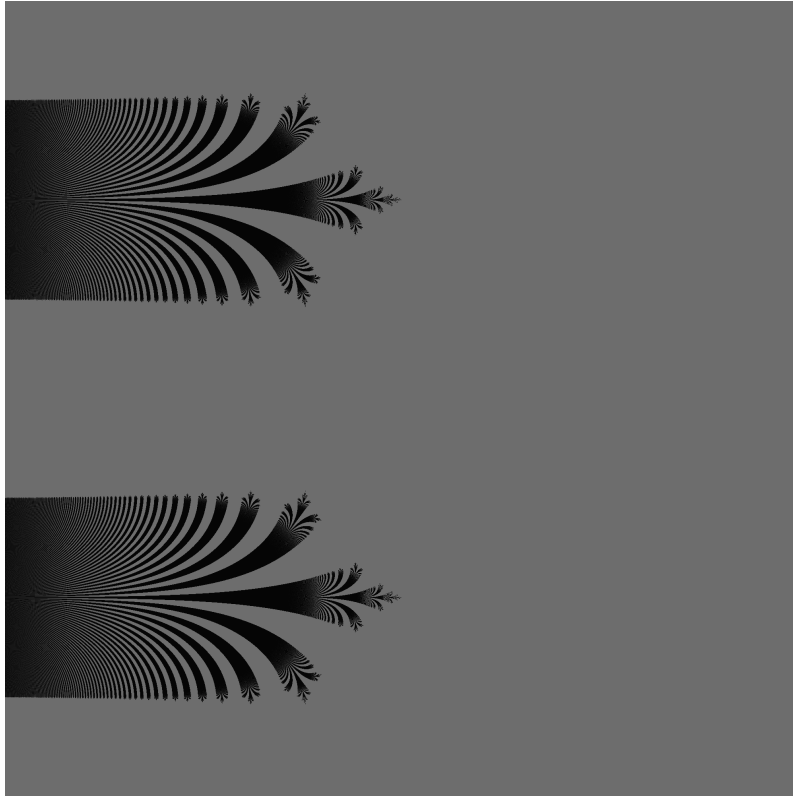
There are many examples of transcendental entire functions having different numbers and/or types of Baker domains. The behaviour of a function in a Baker domain is well understood and a lot of work has been done on functions with Baker domains. For general background and results on Baker domains see, for example, [6], [22], [65]. The first example of a function with a Baker domain was given by Fatou in [35]. Since this function was first studied by Fatou it is sometimes called *Fatou's function* (see [42]). More specifically, for Fatou's function, $f(z) = z + 1 + e^{-z}$, the Fatou set consists of one component which is a Baker domain and contains the open right half-plane $\{z : \operatorname{Re}(z) > 0\}$ (see Figure 1.2 ($F(f)$ in grey)). In the Baker domain the iterates tend to infinity 'slowly' (see also §1.3.1).

1.1.3. The Julia set. The Julia set is a non-empty perfect set (it has no isolated points). A very useful property of the Julia set that we will use in Chapter 5 is the *blowing-up* property, which is described in the following lemma. Let us mention first that for a transcendental entire function f the exceptional set is the set of points with a finite backwards orbit under f and it contains at most one point (see [11, p. 6]).

LEMMA 1.1. *Let f be a transcendental entire function, let K be a compact set which does not meet the exceptional set and let U be an open neighbourhood of $z \in J(f)$. Then there exists $N \in \mathbb{N}$ such that*

$$f^n(U) \supset K, \text{ for all } n \geq N.$$

The Julia set of a function can be the whole complex plane, for example, $J(e^z) = \mathbb{C}$ (see [48]). If it is not the whole complex plane then it has empty interior. The Julia set of a transcendental entire function cannot contain isolated Jordan arcs (see [79]).

FIGURE 1.2. $J(f)$ for $f(z) = z + 1 + e^{-z}$

For many transcendental entire functions the Julia set consists of an uncountable union of disjoint curves each of which joins a finite endpoint to infinity. Such a Julia set is known as a Cantor bouquet; for a precise definition see §1.5.1. Both $f_a(z) = e^z + a$, $a < -1$, and Fatou's function $f(z) = z + 1 + e^{-z}$ have a Cantor bouquet Julia set (the first one contained in the right half-plane and the second one contained in the left half-plane; see Figure 1.1 and Figure 1.2). Using what is called 'Cantor n -bouquets', Devaney and Tangerman showed in [30] that $J(f_a)$ is a Cantor bouquet when $a < -1$ (see also [29]), while in [42] Kotus and Urbanski deduced the same for Fatou's function. This is a different terminology from the one we use in this thesis. We will give more details on this in Section 1.5. The Julia set can also have a very different structure, for example, it can be a connected set containing 'loops' that surround each other. This structure is known as a 'spider's web' and in Section 1.4 we shall discuss it in detail.

1.1.4. Singular values. The set of singular values of f plays an important role in the iteration of transcendental entire functions, for example because of its association with the periodic Fatou components (see [11, Theorem 7]). It is denoted by $\text{Sing}\{f^{-1}\}$ and consists of the critical values and finite asymptotic values of f . A *critical value* is a point $w = f(z)$ with $f'(z) = 0$; the point z is a *critical point*. A finite *asymptotic value* is a point $w \in \mathbb{C}$ such that there exists a curve $\gamma : [0, \infty) \rightarrow \mathbb{C}$ with $\gamma(t) \rightarrow \infty$ and $f(\gamma(t)) \rightarrow w$ as $t \rightarrow \infty$. Note that $f_a(z) = e^z + a$, $a < -1$, has no critical values since $f'_a(z) = e^z \neq 0$ and its only asymptotic value is a . Unlike the exponential family, $f(z) = z + 1 + e^{-z}$ has no finite asymptotic value, although it has infinitely many critical values equal to $2 + 2k\pi i$, $k \in \mathbb{Z}$. Indeed, $f'(z) = 1 - e^{-z}$ and hence for the critical points $2k\pi i$, $k \in \mathbb{Z}$, we have $f(2k\pi i) = 2 + 2k\pi i$, $k \in \mathbb{Z}$.

A very important and very well-studied class of transcendental entire functions is the Eremenko-Lyubich class \mathcal{B} . It consists of transcendental entire functions whose set of singular values is bounded (see [32]). We deduce from the discussion above that $f_a(z) = e^z + a$, $a < -1$, belongs to the class \mathcal{B} whereas $f(z) = z + 1 + e^{-z}$ does not.

1.1.5. The escaping set. Another set which plays a key role in the iteration of transcendental entire functions is the *escaping set*. The escaping set $I(f)$ of f is defined as follows:

$$I(f) = \{z \in \mathbb{C} : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}$$

and was first studied for a general transcendental entire function in [31] by Eremenko who showed that, for any transcendental entire function f , we have $I(f) \cap J(f) \neq \emptyset$, $J(f) = \partial I(f)$ and all the components of $\overline{I(f)}$ are unbounded.

In addition to its interesting properties and its relation to $J(f)$, $I(f)$ has also become a popular object of study because of a famous conjecture that is associated with it (for a discussion see Section 1.3). In [31] Eremenko stated that it is plausible that *all the components of $I(f)$ are unbounded*. This is now known as *Eremenko's conjecture* and still remains an open question.

For functions in the class \mathcal{B} we have that $I(f) \subset J(f)$ by a result of Eremenko and Lyubich in [32]. For $f_a(z) = e^z + a$, $a < -1$, the escaping set consists of the curves in $J(f)$ except for some of their endpoints, as was first shown in [29]. This was the first example of

a function (actually, a class of functions) for which $I(f)$ consists of curves going to infinity. For Fatou's function, $I(f)$ is not a subset of $J(f)$ since all the points in the Fatou set (the Baker domain U) escape. More precisely, the escaping set consists of the Baker domain together with the curves in $J(f)$ except for some of their endpoints (see [72, Example 3]). Moreover, for Fatou's function $I(f)$ is connected since $U \subset I(f) \subset \overline{U} = \mathbb{C}$. In Chapter 4 we will show that in fact we can say much more about $I(f)$ (it actually has the structure of a 'spider's web'; see the discussion in Section 1.4).

Considering the escaping set leads to a different partition of the complex plane, which arises naturally. The complement of the escaping set consists of points that do not escape to infinity under iteration. These points can have bounded orbit, for example, periodic points, in which case we say that they belong to $K(f)$ (see [12] and [51]). Otherwise, they belong to the 'bungee set', $BU(f)$, which consists of the points whose iterates neither escape to infinity nor are bounded. The three sets $I(f)$, $K(f)$ and $BU(f)$ form a partition of the complex plane, each set being unbounded (see [54] for properties of $BU(f)$).

1.2. The maximum modulus: properties and useful results

In this brief section we define the maximum modulus of a transcendental entire function. This is closely related to several rates of escape that we will see in Section 1.3 and we present some results associated with it that we will use in Chapters 2 and 3.

The *maximum modulus* of f on a disc of radius $r > 0$ is defined by

$$M(r, f) = M(r) = \max_{|z|=r} |f(z)|.$$

We denote by $M^n(r, f) = M^n(r)$ the n -th iterate of $M(r, f)$ with respect to r .

One very useful property of the maximum modulus is the convexity property given in the following lemma (see [69, Lemma 2.2]).

LEMMA 1.2. (Rippon and Stallard, 2009) *Let f be a transcendental entire function. Then there exists $R_0 > 0$ such that, for all $r \geq R_0$ and all $c > 1$,*

$$(1.2.1) \quad \log M(r^c) \geq c \log M(r).$$

Sixsmith gave some estimates of the maximum modulus for compositions of functions (see [77, Lemma 2.4]).

LEMMA 1.3. (Sixsmith, 2011) *Suppose that f is a non-constant entire function and g is a transcendental entire function. Then, given $\nu > 1$, there exist $R_1, R_2 > 0$ such that*

$$(1.2.2) \quad M(\nu r, f \circ g) \geq M(M(r, g), f) \geq M(r, f \circ g), \text{ for } r \geq R_1$$

and

$$(1.2.3) \quad M(\nu r, g \circ f) \geq M(M(r, f), g) \geq M(r, g \circ f), \text{ for } r \geq R_2.$$

We also use the following general result related to the maximum modulus (see [71, Theorem 3.1]).

THEOREM 1.4. (Rippon and Stallard, 2014) *Let f be a transcendental entire function. There exists $R = R(f) > 0$ with the property that whenever (a_n) is a positive sequence such that*

$$(1.2.4) \quad a_n \geq R \text{ and } a_{n+1} \leq M(a_n), \text{ for } n \in \mathbb{N},$$

there exists a point $\zeta \in J(f)$ and a sequence (n_j) with $n_j \rightarrow \infty$ such that

$$(1.2.5) \quad |f^n(\zeta)| \geq a_n, \text{ for } n \in \mathbb{N}, \text{ but } |f^{n_j}(\zeta)| \leq M^2(a_{n_j}), \text{ for } j \in \mathbb{N}.$$

The maximum modulus is used to define the *order* $\rho(f)$ and the *lower order* $\lambda(f)$ of a transcendental entire function as follows:

$$\rho(f) = \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} \text{ and } \lambda(f) = \liminf_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r}.$$

REMARK 1.2.1. All functions in the class \mathcal{B} have lower order not less than $1/2$ (see [67, Lemma 3.5]).

A regularity condition related to the maximum modulus which was introduced by Anderson and Hinkkanen in [3] is *log-regularity*. Log-regular functions can be defined in the following way (see [71, Corollary 4.3]) which will be very useful for us when we introduce new regularity conditions related to log-regularity (Chapters 2 and 3).

THEOREM 1.5. (Rippon and Stallard, 2014) *Let f be a transcendental entire function. Then the following are equivalent:*

- (a) f is log-regular;

(b) *there exist $r_0 > 0$ and $c > 0$ such that, for all $k > 1$ and $d = k^c$, we have*

$$M(r)^k \geq M(r)^{kd}, \text{ for } r \geq r_0;$$

(c) *there exist $r_1 > 0$, $k > 1$ and $d > 1$ such that*

$$M(r^k) \geq M(r)^{kd}, \text{ for } r \geq r_1.$$

Roughly speaking, log-regularity prevents transcendental entire functions from behaving very similarly to polynomials over long intervals of values of r . Functions in the class \mathcal{B} as well as functions of finite order and positive lower order are log-regular (see [71, Theorem 5.1] and [39, p. 205] respectively).

We end this section with a very useful result of Clunie (see [26]) which we use in Chapters 2 and 3 in order to construct some examples of entire functions with specific properties of their maximum modulus. Clunie and Kövari proved later a stronger result than the one we use (see [27]).

LEMMA 1.6. (Clunie, 1965) *Let ϕ be a convex increasing function on \mathbb{R} such that $\phi(t) \neq O(t)$ as $t \rightarrow \infty$. Then there exists a transcendental entire function f such that*

$$\log M(e^t, f) \sim \phi(t) \text{ as } t \rightarrow \infty.$$

Lemma 1.6 is a very useful tool that allows us to obtain transcendental entire functions with certain properties by first constructing real, increasing, convex functions.

1.3. Rates of escape

We will see in this section that points in the escaping set can escape to infinity at different rates. The rates of escape that we consider in this thesis are uniform rates of escape. We say that a set $U \subset I(f)$ has a *uniform rate of escape* if for every $R > 0$ there exists $N \in \mathbb{N}$ such that $|f^n(z)| \geq R$, for all $n \geq N$ and all $z \in U$. In Chapters 4 and 5 we consider sets defined by

$$I(f, (a_n)) = \{z \in \mathbb{C} : |f^n(z)| \geq a_n, \text{ for } n \in \mathbb{N}\},$$

where (a_n) is a positive sequence such that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. In particular, these sets are our most important tool in the proofs of the main results in Chapters 4 and 5.

Points in a Baker domain escape to infinity relatively slowly (see [68, Theorem 1.2]) and so they can only belong to $I(f, (a_n))$ if the sequence (a_n) does not tend to infinity too quickly. Throughout the thesis we focus on sets with a uniform rate of escape, their properties, structure and important role in proving results which sometimes have no obvious connection with these sets.

1.3.1. The fast escaping set. Much work on the iteration of transcendental entire functions during the last two decades has been motivated by Eremenko's conjecture that all the components of the escaping set are unbounded. Significant progress has been made on the conjecture by Rippon and Stallard who proved that $I(f)$ has at least one unbounded component (see [70, Theorem 1]). In order to do this, they considered a subset of the escaping set known as the *fast escaping set*, $A(f)$. This set was introduced by Bergweiler and Hinkkanen in [15]. We will use the definition given by Rippon and Stallard in [68]; see also [20]:

$$\begin{aligned} A(f) &= \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\} \\ &= \bigcup_{\ell \in \mathbb{N}} f^{-\ell}(A_R(f)), \end{aligned}$$

where $R > 0$ is large enough to ensure that $M(r) > r$ for $r \geq R$ and $A_R(f)$ is defined as follows:

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}.$$

In the same paper they showed that $A(f)$ has properties similar to the properties of $I(f)$ listed in §1.1.5. (Some of these results were shown in [15].) Most importantly they showed that *all* the components of the fast escaping set are unbounded (see [70, Theorem 1] and [68, Theorem 1.1]). Note that $A(f)$ is independent of the choice of R as long as $R > 0$ is such that $M(r) > r$ for $r \geq R$.

Roughly speaking, $A(f)$ consists of the points that escape to infinity 'as quickly as possible'. Note that the fast escaping set is a strict subset of the escaping set since for any transcendental entire function there are always points that escape arbitrarily slowly to infinity (see [72, Theorem 1]).

A negative answer to the strong form of Eremenko's conjecture, that every point in $I(f)$ can be joined with infinity by a curve in $I(f)$, was given in [74]. In the same paper

the authors considered finite compositions of functions of finite order in the class \mathcal{B} , and proved, using some ideas related to the fast escaping set, that for this class of functions the conclusion of the strong form of the conjecture holds.

THEOREM 1.7. (Rottenfußer, Rückert, Rempe and Schleicher, 2009) *Let f be a finite composition of functions of finite order in the class \mathcal{B} and let $z_0 \in I(f)$. Then z_0 can be connected to ∞ by a simple curve $\gamma \subset I(f)$ such that $f^n \rightarrow \infty$ as $n \rightarrow \infty$ uniformly on γ .*

Rempe, Rippon and Stallard showed that the escaping points given by Theorem 1.7 are actually fast escaping for this class of functions, apart from some of the endpoints of the curves.

THEOREM 1.8. (Rempe, Rippon and Stallard, 2010) *Let f be as in Theorem 1.7. Then all the points on γ , except possibly for the endpoint, belong to $A(f)$.*

Recall from §1.1.5 that, for $f_a(z) = e^z + a$, $a < -1$, $I(f_a)$ consists of the curves in $J(f_a)$ except for some of their endpoints. Since such a function has order 1 and is in the class \mathcal{B} we deduce from Theorem 1.8 that actually $A(f_a)$ consists of the curves in $J(f_a)$ except for some of their endpoints. Note that since $A(f_a) \subsetneq I(f_a)$ there must be endpoints of the curves that escape but not ‘fast’. The fact that the points in the curves escape fast for this family of functions was first shown in [76].

Functions outside the class \mathcal{B} can have fast escaping points that belong to the Fatou set. In fact, the only Fatou components that can be fast escaping are wandering domains (see [15, Lemma 4]). In particular, as we mentioned earlier, Baker domains are never fast escaping (see [68, Theorem 1.2]).

Hence for Fatou’s function we also have $A(f) \subset J(f)$. In particular, $A(f)$ consists of the curves in $J(f)$ except for some of their endpoints (see [72, Example 3]).

As for $J(f)$ and $I(f)$, the fast escaping set can also have a very different structure than the one described above, which is the case when it is a ‘spider’s web’ (see Section 1.4).

The set $A(f)$ also has other nice properties (described in [68]) and plays a key role in the iteration of transcendental entire functions and, more widely, in the iteration of quasiregular mappings and transcendental self-maps of the punctured plane (see [18] and [43]). Thus it is useful to be able to identify points that are fast escaping.

1.3.2. The quite fast escaping set. In [68, Theorem 2.7], it is shown that points that eventually escape faster than the iterates of the function $r \mapsto \varepsilon M(r)$, $r > 0$, are actually fast escaping. It is natural to ask whether the above function can be replaced by a smaller function. In this context, Rippon and Stallard introduced the *quite fast escaping set* $Q(f)$ in [71]. Let $Q_\varepsilon(f)$, $\varepsilon \in (0, 1)$, be defined to be the set

$$Q_\varepsilon(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_\varepsilon^n(R), \text{ for } n \in \mathbb{N}\},$$

where $\mu_\varepsilon = M(r)^\varepsilon$, and $R > 0$ is large enough to ensure that $\mu_\varepsilon(r) > r$ for $r \geq R$. The definition of $Q_\varepsilon(f)$ is independent of the choice of R with the property that $\mu_\varepsilon(r) > r$ for $r \geq R$. Then the quite fast escaping set is defined as follows:

$$Q(f) = \bigcup_{0 < \varepsilon < 1} Q_\varepsilon(f).$$

The set $Q(f)$ arises naturally in complex dynamics and so it is of interest to establish when $Q(f)$ is equal to $A(f)$. Although Rippon and Stallard were the first to define $Q(f)$, points that belong to $Q(f)$ were used earlier in results concerning the Hausdorff measure and Hausdorff dimension of the escaping set and the Julia set of some classes of functions (see [17] and [55]). Rippon and Stallard showed that $Q(f) = A(f)$ for many classes of functions, such as *log-regular* functions, but they also constructed examples where $Q(f) \neq A(f)$.

As we mentioned in Section 1.2, functions in the class \mathcal{B} are log-regular and so for such functions we have $Q(f) = A(f)$. In particular, this is the case for $f_a(z) = e^z + a$, $a < -1$.

1.3.3. Generalising the quite fast escaping set. Looking for functions even smaller than μ_ε we introduce a family of sets defined as follows:

$$Q_m(f) = \{z : \exists \varepsilon \in (0, 1), \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_{m,\varepsilon}^n(R), \text{ for } n \in \mathbb{N}\},$$

where $\mu_{m,\varepsilon}$, $\varepsilon \in (0, 1)$, is defined by the equation

$$\log^m \mu_{m,\varepsilon}(r) = \varepsilon \log^m M(r), \quad m \in \mathbb{N},$$

whenever the right-hand side of the equality is well defined, and whenever there exists $R = R(f, \varepsilon) > 0$ such that $\mu_{m,\varepsilon}(r) > r$ for $r \geq R$; in particular, we always have $Q_m(f) \subset I(f)$.

Note that it is not always true that such an $R = R(f, \varepsilon)$ exists (see Chapters 2 and 3). The sets $Q_m(f)$ as defined above are again independent of the choice of R .

This family generalises the quite fast escaping set. Indeed, in the case where $m = 1$ we obtain the quite fast escaping set $Q(f)$; that is, $Q_1(f) = Q(f)$. Note that for $m = 1$ there always exists $R > 0$ such that $\mu_{1,\varepsilon}(r) > r$ for $r \geq R$ (unlike the general case; see also Section 3.2). Since for $\varepsilon \in (0, 1)$ we have $\mu_{m,\varepsilon}(r) < \mu_{1,\varepsilon}(r) < M(r)$, we deduce that for any $m \geq 2$ and for r large enough

$$(1.3.1) \quad A(f) \subset Q(f) \subset Q_m(f) \subset I(f).$$

The sets $Q_m(f)$ also have properties similar to $I(f)$ (see Theorem 3.5).

It is again interesting to know when $Q_m(f) = A(f)$, $m \in \mathbb{N}$. Hence we introduce new regularity conditions that are sufficient for $Q_m(f)$ to be equal to $A(f)$. Our main result gives a large class of functions for which $Q_m(f) = A(f)$, $m \in \mathbb{N}$; precisely the functions of finite order and positive lower order (Theorem 3.1). Functions in the class \mathcal{B} of finite order belong to this class of functions (for example, $f_a(z) = e^z + a$). Finite compositions of such functions were considered in Theorems 1.7 and 1.8. A stronger version of the result for $Q_m(f)$ for the case where $m = 2$ is proved in Chapter 2 (Theorem 2.1).

Although we provide a large class of functions for which $Q_m(f) = A(f)$, for all $m \in \mathbb{N}$, in Chapter 2 we give an example of a function for which $Q_2(f) \neq A(f)$ whereas $Q(f) = A(f)$. In order to construct this function as well as two other examples of functions with specific properties related to our newly introduced regularity conditions (Examples 2.4.1 and 3.5.1), we make use of Lemma 1.6.

1.4. Spiders' webs

To obtain a better understanding of the possible structure of the escaping set and the fast escaping set Rippon and Stallard showed that, for several families of transcendental entire functions, $A_R(f)$ has a particular structure defined as follows (see [68]).

DEFINITION 1.9. A set $E \subset \mathbb{C}$ is an *(infinite) spider's web* if E is connected and there exists a sequence (G_n) of bounded simply connected domains, $n \in \mathbb{N}$, with $G_n \subset G_{n+1}$, for $n \in \mathbb{N}$, $\partial G_n \subset E$, for $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} G_n = \mathbb{C}$.

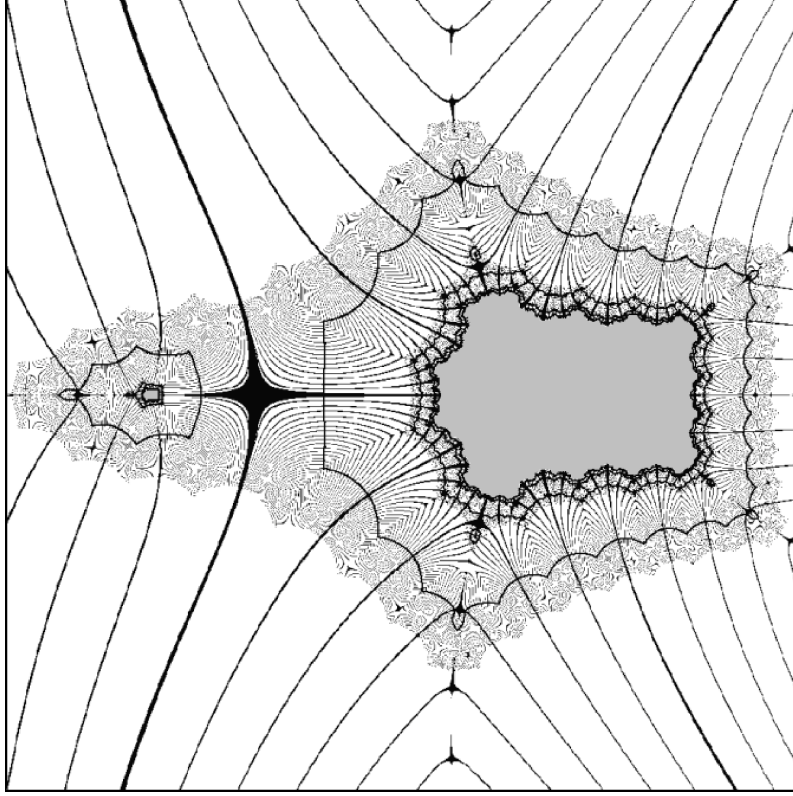


FIGURE 1.3. An $A_R(f)$ spider's web for $f(z) = \frac{1}{2}(\cos z^{1/4} + \sin z^{1/4})$

From the definition we see that a spider's web has a structure where there are loops (∂G_n) that surround each other. Note, however, that the choice of these loops is not always obvious. For example, the whole complex plane is a spider's web with many different choices of the domains G_n .

Rippon and Stallard also proved that if $I(f)$ *contains* a spider's web then it *is* a spider's web (see [66, Lemma 4.5]). Hence, whenever we have an $A_R(f)$ spider's web we also have an $I(f)$ spider's web. Many examples are now known of functions for which $I(f)$ is a spider's web (see, for example, [47], [68], [77]). Specific examples of such functions are $z \mapsto \frac{1}{2}(\cos z^{1/4} + \sin z^{1/4})$ (see Figure 1.3) and $z \mapsto \cos z + \cosh z$.

The spider's web structure is significantly different from the structure of Cantor bouquets mentioned in Sections 1.1 and 1.3. Note also that for a function in the class \mathcal{B} , the

set $A_R(f)$ cannot be a spider's web (see [68, Theorem 1.8]). Whenever $A_R(f)$ is a spider's web, f has many strong dynamical properties (see [52] and [68]).

Spiders' webs have surprising connections with two of the most famous open questions in the theory of iteration of transcendental entire functions. First, note that when $I(f)$ is a spider's web then it is connected and hence Eremenko's conjecture holds. Indeed, in this case $I(f)$ consists of one unbounded component. Moreover, Rippon and Stallard showed that if $A_R(f)$ is a spider's web then f has no unbounded Fatou components ([68, Theorem 1.5]). This implies that, if $A_R(f)$ is a spider's web then the conclusion of Baker's conjecture holds. Baker's conjecture states that entire functions of order less than $1/2$ have no unbounded Fatou components. This remains an open question (see [39] and [49]).

Until now, there was only one known example of a function (a very complicated infinite product) for which $I(f)$ is a spider's web but $A(f)$ is not a spider's web (see [66, Theorem 1.1]). In Chapter 4 we give the first simple example of a function for which $I(f)$ is a spider's web but $A(f)$ is not. This example is Fatou's function $f(z) = z + 1 + e^{-z}$. As mentioned earlier, for this function $I(f)$ is connected and consists of the Baker domain and the curves in $J(f)$ except for some of their endpoints, whereas $A(f)$ consists of the curves in $J(f)$ except for some of their endpoints and it is not connected. Since this case is different from the classes of functions for which $A_R(f)$ is a spider's web, studied earlier, new techniques needed to be developed and another set defined by a uniform rate of escape was used instead of $A_R(f)$. In particular we considered $I(f, (a_n))$, where $a_n = (n + 6)/2$ (see Chapter 4). In fact, we show that a whole class of functions which have a Baker domain and similar dynamical behaviour to f has the same property. In particular, this class includes a function that was studied first by Bergweiler in [10], namely, $f(z) = 2z + 2 - \log 2 - e^z$. Nicks and Sixsmith have asked whether the quasiregular map they constructed as an analogue of Fatou's function has the same property (see [50, Remark (2), p.22]), which remains an open question.

General theorems which imply that $I(f)$ or $F(f) \cup A(f)$ is a spider's web are given in Chapter 4. These theorems or variations thereof could potentially be used to obtain similar results for other classes of functions as well (see the discussion in Chapter 6).

Finally, in Section 5.5 we give a different technique for constructing spiders' webs which uses the blowing-up property of the Julia set and the set $A_R(f)$.

In addition to the fact that spiders' webs give us important information about the structure of $I(f)$ and $A(f)$ and are related to two famous conjectures, they also have interesting connections to the endpoints of the curves in $J(f)$. We described this connection first in [34] and we discuss it further in the following section.

1.5. Cantor bouquets and endpoints

1.5.1. Cantor bouquets. As we mentioned in Section 1.1, the Julia set for several transcendental entire functions is a 'Cantor bouquet'. In the examples we gave we described this structure as an uncountable union of disjoint curves each of which joins a finite endpoint to infinity. We now give a precise definition of a Cantor bouquet and for this we first need to define a straight brush. We follow the definitions given in [8] (which are based on terminology introduced in [1]). Detailed information on straight brushes and their properties can be found in [1].

DEFINITION 1.10. A subset B of $[0, +\infty) \times (\mathbb{R} \setminus \mathbb{Q})$ is called a *straight brush* if the following properties are satisfied:

- (a) The set B is a closed subset of \mathbb{R}^2 .
- (b) For every $(x, y) \in B$ there exists $t_y \geq 0$ such that $\{t : (t, y) \in B\} = [t_y, +\infty)$. The set $[t_y, +\infty) \times \{y\}$ is called the *hair* attached at y and the point (t_y, y) is called its *endpoint*.
- (c) The set $\{y : (x, y) \in B \text{ for some } x\}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for every $(x, y) \in B$ there exist two sequences of hairs attached respectively at $\beta_n, \gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\beta_n < y < \gamma_n, \beta_n, \gamma_n \rightarrow y$ and $t_{\beta_n}, t_{\gamma_n} \rightarrow t_y$ as $n \rightarrow \infty$.

Although we referred to endpoints in the previous sections, in (b) we see their precise definition. The set of endpoints of a straight brush B is dense in B (see [1, Corollary 2.5] and [2, p. 7]). The final thing we need before defining Cantor bouquets is a stronger version of homeomorphicity. We say that two subsets of \mathbb{R}^2 , X and Y , are *ambiently homeomorphic* if there exists a homeomorphism (i.e. a continuous, bijective function whose inverse is also continuous) $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(X) = Y$.



FIGURE 1.4. Part of the Julia set of $g(z) = e^{-1}ze^{-z}$ in $[-5, 4] \times [-7, 2]$

DEFINITION 1.11. A *Cantor bouquet* is any subset of the plane that is ambiently homeomorphic to a straight brush.

Note that the only points of a Cantor bouquet that are accessible from its complement are the endpoints (see [2, Theorem 2.8(b)]). A point z_0 in the Cantor bouquet is accessible if there is a continuous curve $\gamma : [0, \infty) \rightarrow \mathbb{C}$ for which $\gamma(t)$ lies in the complement of the Cantor bouquet for all t and $\lim_{t \rightarrow \infty} \gamma(t) = z_0$.

We have already given specific examples of functions whose Julia set is a Cantor bouquet ($f_a(z) = e^z + a, a < -1$, and $f(z) = z + 1 + e^{-z}$). Barański, Jarque and Rempe described a large class of functions for which $J(f)$ is a Cantor bouquet. In order to state their result we will first define hyperbolic functions.

DEFINITION 1.12. A transcendental entire function is called *hyperbolic* if the postsingular set defined by

$$\mathcal{P}(f) = \overline{\bigcup_{j \geq 0} f^j(\text{Sing}(f^{-1}))}$$

is a compact subset of the Fatou set $F(f)$.

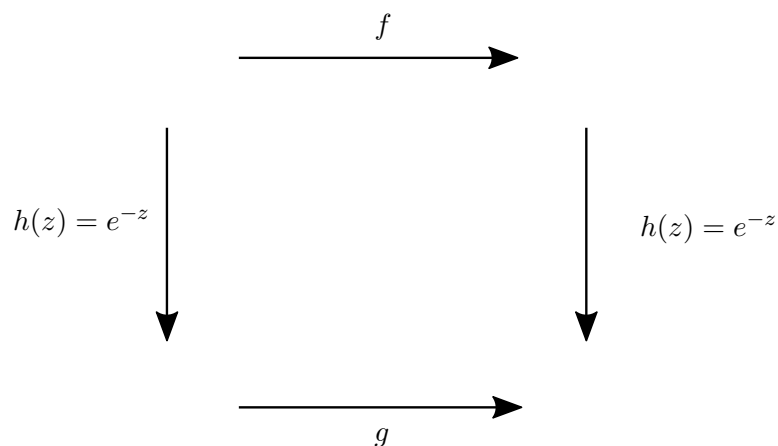
(Recall that we defined $\text{Sing}(f^{-1})$ in §1.1.4.)

A function f is of *disjoint type* if it is hyperbolic and $F(f)$ is connected. This implies that for functions of disjoint type the Fatou set consists of one component which is an immediate attracting basin. We have the following result.

THEOREM 1.13. (Barański, Jarque and Rempe, 2012) *Let f be a disjoint-type function which has finite order or can be written as a finite composition of finite-order functions in the class \mathcal{B} . Then $J(f)$ is a Cantor bouquet.*

A simple example of a class of functions of disjoint type is given by $f_a(z) = e^z + a$, $a < -1$. Another example of a function which belongs to this class is $g(z) = e^{-1}ze^{-z}$. Indeed, g is of order 1 and 0 is an attracting fixed point for g . Also it is not hard to see that the only two singularities of g^{-1} , namely 0 and e^{-2} , lie in the attracting basin of 0, which is the only Fatou component. Hence $J(g)$ is a Cantor bouquet (see Figure 1.4). Note now that Fatou's function is a lift of g by $h(z) = e^{-z}$. In other words, $h \circ f = g \circ h$ (see Figure 1.5). Hence $J(f)$ is also a Cantor bouquet since the exponential map is a local homeomorphism.

1.5.2. Pinched Cantor bouquets. In the case of a Cantor bouquet, every hair has a different endpoint. We will now consider a more general topological structure which allows different hairs to share the same endpoint. We call such a subset of \mathbb{R}^2 a *pinched Cantor bouquet*. It is defined to be the quotient of a Cantor bouquet by a closed equivalence relation defined on its endpoints. We say that an equivalence relation R on a topological space X is closed when the quotient map $X \rightarrow X/R$ is closed. A pinched Cantor bouquet again consists of curves and their endpoints (see [46]). The largest known class of functions for which the Julia set is a pinched Cantor bouquet was given in [46] and is the following.

FIGURE 1.5. Fatou's function as a lift of g

THEOREM 1.14. (Mihaljević-Brandt, 2012) *Let $f = f_1 \circ \dots \circ f_n$ be a strongly subhyperbolic map, where f_i is a transcendental entire function of finite order with a bounded set of singular values. Then $J(f)$ is a pinched Cantor bouquet.*

A transcendental entire function f is called *strongly subhyperbolic* if $\mathcal{P}(f) \cap F(f)$ is compact, $\mathcal{P}(f) \cap J(f)$ is finite, $J(f)$ does not contain any asymptotic value of f and the local degree of f at the points in $J(f)$ is uniformly bounded by some finite constant.

An example of a family of functions which are strongly subhyperbolic is given by $f_a(z) = e^z + a$, whenever f_a has an attracting cycle of Fatou components. However, these functions are also hyperbolic and their Julia set was shown to be a pinched Cantor bouquet already in [59]. Note that this includes the case where $a < -1$ where $J(f_a)$ is a simple Cantor bouquet. Whenever $f_a(z) = e^z + a$, $a \in F(f_a)$, and f_a has a parabolic cycle of Fatou components f_a is not strongly hyperbolic but its Julia set is a pinched Cantor bouquet as stated in [61] (the detailed proof of this result is work in progress by Mashaël Alhamd).

1.5.3. Background from topology. In order to discuss some very interesting properties of the endpoints of Cantor bouquets (and pinched Cantor bouquets) we first need some definitions from topology.

DEFINITION 1.15. A topological space X is *totally disconnected* if the connected components of X are one-point sets.

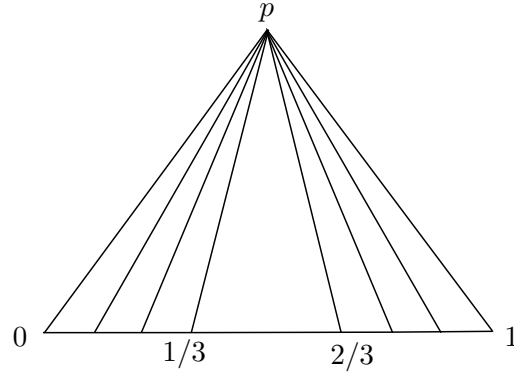


FIGURE 1.6. Construction of Cantor's leaky tent

An example of a totally disconnected set is given by the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$. A stronger topological property is the following.

DEFINITION 1.16. Let X be a topological space. Two points $a, b \in X$ are *separated* (in X) if there is an open and closed subset $U \subset X$ with $a \in U$ and $b \notin U$. If any two points of X are separated, then we say that X is a *totally separated* space.

A totally separated space is totally disconnected (in fact, $\mathbb{R} \setminus \mathbb{Q}$ is also totally separated) but the converse is not always true. An example of a space which is totally disconnected but not totally separated is given by the *Knaster-Kuratowski fan*, or as it is also known, *Cantor's leaky tent* (see [41, Exemple α] and [78, p. 145]). In order to construct it we take the Cantor set C situated on $[0, 1]$ and we join every point $c \in C$ to the point $p = (1/2, 1/2)$ by a line segment L_c (see Figure 1.6). Then for the points of C that are endpoints of deleted intervals we keep only the points (x, y) of L_c for which $y \in \mathbb{Q}$. For the rest of the points of C we only keep the points of L_c for which $y \in \mathbb{R} \setminus \mathbb{Q}$. The union of all these points is a subset of \mathbb{R}^2 that is totally disconnected but when we add p we have a connected set. In fact it was the first example of a space with such a property. We will discuss examples in complex dynamics with similar properties below.

A subspace of a totally disconnected (totally separated) space is totally disconnected (totally separated). In this thesis we use the following form of separation which was given in [2] (applied to $\hat{\mathbb{C}}$ instead of \mathbb{C}).

LEMMA 1.17. (Alhabib and Rempe-Gillen, 2015) *Let $X \subset \hat{\mathbb{C}}$. Then x and y are separated in X if and only if there is a closed and connected set $\Delta \subset \hat{\mathbb{C}} \setminus X$ such that x and y belong to different components of $\hat{\mathbb{C}} \setminus \Delta$.*

1.5.4. Endpoints. As we mentioned in Section 1.1, the exponential family $f_a(z) = e^z + a, a \in \mathbb{C}$, is probably the most well-studied family of transcendental entire functions. In §1.5.1 we defined the endpoints of hairs in a straight brush. In this subsection we focus on functions in the exponential family, not necessarily having a Cantor bouquet Julia set. We say that a point $z_0 \in \mathbb{C}$ is on a *hair* if there exists an arc $\gamma : [-1, 1] \rightarrow I(f_a)$ such that $\gamma(0) = z_0$. A point $z_0 \in \mathbb{C}$ is an *endpoint* if z_0 is not on a hair and there is an arc $\gamma : [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = z_0$ and $\gamma(t) \in I(f_a)$ for all $t > 0$.

For the case where $a < -1$ a lot of work has been done on the endpoints of the curves in $J(f_a)$ and there are several interesting results known, including Karpińska's celebrated result that the Hausdorff dimension of these points is 2 even though the Hausdorff dimension of the curves is 1 ([40]). Mayer showed in [44] that the set of *endpoints*, $E(f_a)$, for $f_a(z) = e^z + a, a < -1$, is totally separated but the union of all endpoints with infinity is a connected set.

THEOREM 1.18. (Mayer, 1990) *Let $f_a(z) = e^z + a, a < -1$. Then $E(f_a)$ is totally separated but $E(f_a) \cup \{\infty\}$ is connected.*

For a topological space X we say that p is an *explosion point* of X if X is connected but $X \setminus \{p\}$ is totally separated. Following the terminology used in [2], we say that infinity is an explosion point for the set of endpoints $E(f_a)$, when $f_a(z) = e^z + a, a < -1$.

Recently, Alhabib and Rempe-Gillen showed in [2] that Mayer's result holds for the *escaping* endpoints of $J(f_a)$, (that is, the endpoints that belong to the escaping set) for three classes of functions in the exponential family. Let $\tilde{E}(f)$ denote the set of escaping endpoints, that is, $\tilde{E}(f) = E(f) \cap I(f)$. Then the following is true.

THEOREM 1.19. (Alhabib and Rempe-Gillen, 2015) *Let $f_a(z) = e^z + a, a \in \mathbb{C}$. Suppose that a satisfies one of the following conditions.*

- (a) $a \in F(f_a)$;
- (b) a is on a hair;

(c) a is an endpoint.

Then ∞ is an explosion point for $\tilde{E}(f_a)$.

Moreover, in the same paper the authors showed that for case (a) Mayer's result on the set of all endpoints holds, generalising his original result.

THEOREM 1.20. (Alhabib and Rempe-Gillen, 2015) *Let $f_a(z) = e^z + a$, $a \in F(f_a)$. Then ∞ is an explosion point for $E(f_a)$.*

In fact, they observed that Theorem 1.20 is true for any function whose Julia set is a Cantor bouquet, such as those satisfying the hypotheses of Theorem 1.13 (This result uses [2, Proposition 2.5] that was originally proved in [24] and [25].)

THEOREM 1.21. (Alhabib and Rempe-Gillen, 2015) *Let f be a transcendental entire function such that $J(f)$ is a Cantor bouquet. Then ∞ is an explosion point for $E(f)$.*

1.5.5. Non-escaping endpoints. Following the results of Mayer and Alhabib and Rempe-Gillen, it is natural to ask whether the same property holds for the set of *non-escaping* endpoints for the functions they have considered. Here non-escaping endpoints are, by definition, the endpoints that do not belong to the escaping set, and we denote this set by $\hat{E}(f)$. So

$$\hat{E}(f) = E(f) \setminus I(f) = E(f) \setminus \tilde{E}(f).$$

The non-escaping endpoints belong either to $K(f)$ or to $BU(f)$ (defined in §1.1.5). The set of non-escaping endpoints has been studied regarding its dimension (see the discussion in Section 5.2) but, to the best of our knowledge, there was no result so far regarding its topology. In Chapter 5, we show that for $f_a(z) = e^z + a$, $a \in F(f_a)$, the non-escaping endpoints cannot ‘explode’ since their union with infinity is itself a totally separated set. Since our result uses Theorem 1.20, this was the natural class of functions to consider within the exponential family. In order to prove our result we consider again a uniformly escaping set $I(f_a, (a_n))$ where (a_n) is a positive, increasing sequence such that $I(f_a, (a_n)) \subset A_R(f_a)$, for some $R > 0$.

The first result of this nature on non-escaping endpoints is presented in Section 5.2 and concerns Fatou's function (see also [34]). More specifically, using the fact that for Fatou's

function $I(f)$ is a spider's web we show in Section 5.2 that the union of non-escaping endpoints of $J(f)$ with infinity is a totally separated set. Using a different technique (similar to the one used for the exponential family) we give a stronger result for Fatou's function in Section 5.4.

CHAPTER 2

Fast escaping points of entire functions: a new regularity condition

2.1. Introduction

In this chapter and Chapter 3 we consider the sets $Q_m(f)$ introduced in §1.3.3. Recall that the sets $Q_m(f)$ are generalisations of the quite fast escaping set $Q(f)$. In this chapter we focus on the set $Q_2(f)$. Hence we consider

$$(2.1.1) \quad \mu_{2,\varepsilon}(r) = \exp((\log M(r))^\varepsilon), \quad 0 < \varepsilon < 1,$$

and we set

$$(2.1.2) \quad Q_2(f) = \{z : \exists \varepsilon \in (0, 1) \text{ and } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_{2,\varepsilon}^n(R), \text{ for } n \in \mathbb{N}\},$$

where $R > 0$ is such that $\mu_{2,\varepsilon}(r) > r$ for $r \geq R$.

Unlike the functions $\mu_\varepsilon(r) = M(r)^\varepsilon$ that were introduced in earlier papers, for $\mu_{2,\varepsilon}$ we do not know a priori that, for any given transcendental entire function, $\mu_{2,\varepsilon}(r) > r$ for r large enough. In fact, for some slowly growing functions f , $Q_2(f)$ is not defined. However, there is a large class of functions for which $Q_2(f)$ is defined, and, as we showed in (1.3.1), for such functions we have $A(f) \subset Q(f) \subset Q_2(f) \subset I(f)$. As we have already mentioned it is very useful to be able to find points that are fast escaping. Hence we seek to identify functions for which $Q_2(f) = A(f)$. In this chapter we give a large class of functions for which this is true.

THEOREM 2.1. *Let $f = f_1 \circ f_2$ be a composition of transcendental entire functions, where f_1 has finite order and positive lower order. Then $Q_2(f) = A(f)$.*

An immediate consequence of Theorem 2.1 is the following:

COROLLARY 2.2. *We have $Q_2(f) = A(f)$ whenever $f = f_1 \circ f_2$ is a composition of transcendental entire functions and f_1 satisfies one of the following:*

(a) *there exist $A, B, C, r_0 > 1$ such that*

$$A \log M(r, f_1) \leq \log M(Cr, f_1) \leq B \log M(r, f_1), \text{ for } r \geq r_0;$$

(b) *$f_1 \in \mathcal{B}$ and is of finite order.*

In fact, functions of type (a) were studied by Bergweiler and Karpińska in [16] where it was shown that they are of finite order and positive lower order. Finite compositions of functions of type (b) were considered in [74] and [28], as mentioned in §1.3.1.

The proof of Theorem 2.1 is in three steps. We first introduce a new regularity condition as follows:

A transcendental entire function f is *strongly log-regular* if, for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that

$$(2.1.3) \quad \log M(r^k) \geq (k \log M(r))^{1/\varepsilon}, \text{ for } r > R.$$

Using (2.1.1) we see that for each $\varepsilon \in (0, 1)$, (2.1.3) is equivalent to

$$(2.1.4) \quad \mu_{2,\varepsilon}(r^k) \geq M(r)^k, \text{ for } r > R,$$

which, since $\mu_{2,\varepsilon}(r) < M(r)^\varepsilon$, implies that

$$(2.1.5) \quad M(r^k)^\varepsilon \geq M(r)^k, \text{ for large } r,$$

or equivalently, there exist $R_0 > 0, k, d > 1$ such that

$$(2.1.6) \quad M(r^k) \geq M(r)^{kd}, \text{ for } r > R_0.$$

By Theorem 1.5, this latter condition is equivalent to log-regularity that was used in [71] as a sufficient condition for $Q(f) = A(f)$. Log-regularity was discussed in Section 1.2. The name strong log-regularity arises from the fact that strong log-regularity implies log-regularity.

In Section 2.2, we give the first two steps of the proof of Theorem 2.1. In the first step, we show that strong log-regularity is a sufficient condition for $Q_2(f) = A(f)$. In the second step, we prove that any transcendental entire function of finite order and positive lower order is strongly log-regular.

The last step of the proof is given in Section 2.3 where we show that strong log-regularity is preserved under the composition of two transcendental entire functions where the first function of the composition is strongly log-regular.

Finally, in the last section of this chapter, we construct two functions. The first is an example of a strongly log-regular function with zero lower order and positive, finite order and the second is a function for which $Q_2(f) \neq A(f)$ whereas $Q(f) = A(f)$.

2.2. Sufficient conditions for $Q_2(f) = A(f)$

In this section we give the first two steps of the proof of Theorem 2.1. We first prove the following result about strongly log-regular functions:

THEOREM 2.3. *Let f be a transcendental entire function which is strongly log-regular. Then $Q_2(f) = A(f)$.*

PROOF. Clearly $A(f) \subset Q_2(f)$, as noted earlier. Suppose now that $z \in Q_2(f)$. Then (2.1.2) implies that there exist $\varepsilon \in (0, 1)$ and $\ell \in \mathbb{N}$ such that

$$(2.2.1) \quad |f^{n+\ell}(z)| \geq \mu_{2,\varepsilon}^n(R), \quad \text{for } n \in \mathbb{N},$$

where $R > 0$ is such that $\mu_{2,\varepsilon}(r) > r$ for $r \geq R$. As f is strongly log-regular it satisfies (2.1.4) and so there exist $R_0 > R$ and $k > 1$ such that, for $r > R_0$,

$$(2.2.2) \quad \mu_{2,\varepsilon}(r^k) \geq M(r)^k, \quad \text{for } r > R_0.$$

By applying (2.2.2) twice we obtain

$$\mu_{2,\varepsilon}^2(r^k) = \mu_{2,\varepsilon}(\mu_{2,\varepsilon}(r^k)) \geq \mu_{2,\varepsilon}((M(r))^k) \geq (M(M(r)))^k, \quad \text{for } r > R_0$$

since $\mu_{2,\varepsilon}(r^k) > R$. By applying (2.2.2) repeatedly in this way we obtain that

$$(2.2.3) \quad \mu_{2,\varepsilon}^n(r^k) \geq (M^n(r))^k \geq M^n(r), \quad \text{for } r > R_0.$$

But $M^n(r) \rightarrow \infty$ as $n \rightarrow \infty$ for $r \geq R$ and so there exists $n_0 \in \mathbb{N}$ such that $M^{n_0}(R) \geq R^k$ and hence, (2.2.1) and (2.2.3) imply that

$$|f^{n+n_0+\ell}(z)| \geq \mu_{2,\varepsilon}^{n+n_0}(R) \geq M^{n+n_0}(R^{1/k}) \geq M^n(R),$$

and the result follows. \square

We now show that all functions of finite order and positive lower order are strongly log-regular.

THEOREM 2.4. *Let f be a transcendental entire function of finite order and positive lower order. Then f is strongly log-regular and hence $Q_2(f) = A(f)$.*

PROOF. Let f be a transcendental entire function of finite order and positive lower order. Then there exist $0 < q < p$ such that

$$(2.2.4) \quad e^{r^q} \leq M(r) \leq e^{r^p}, \text{ for sufficiently large } r,$$

or equivalently

$$r^q \leq \log M(r) \leq r^p.$$

So, for each $\varepsilon \in (0, 1)$ and sufficiently large r ,

$$(2.2.5) \quad (\log M(r^k))^\varepsilon \geq (r^{qk})^\varepsilon = r^{\varepsilon qk}.$$

It follows from (2.2.4) that, for $k > p/(q\varepsilon)$, there exists $R > 0$ such that, for $r > R$,

$$(2.2.6) \quad r^{\varepsilon qk} \geq kr^p \geq k \log M(r),$$

so (2.1.3) is satisfied, by (2.2.5) and (2.2.6). \square

2.3. Composition and strong log-regularity

In this section we complete the proof of Theorem 2.1 by showing that the composition of two transcendental entire functions, where the first function of the composition is strongly log-regular, is a strongly log-regular function. This implies that if f is strongly log-regular then the n -th iterate f^n is strongly log-regular as well.

THEOREM 2.5. *Let f_1 and f_2 be transcendental entire functions and suppose f_1 is strongly log-regular. Then $g = f_1 \circ f_2$ is strongly log-regular.*

PROOF. Let f_1 be strongly log-regular, that is, for any $\varepsilon \in (0, 1)$ there exist $R > 0$ and $k > 1$ such that

$$(2.3.1) \quad (\log M(r^k, f_1))^\varepsilon \geq k \log M(r, f_1), \quad \text{for } r \geq R,$$

and let f_2 be any transcendental entire function.

Given $\varepsilon' \in (0, 1)$ we take $\varepsilon = \frac{2}{3}\varepsilon'$. Then there exist $R > 0$ and $k > 1$ such that (2.3.1) holds with this ε . Now take $\nu = k^{1/2}$ and put $k' = \nu k = k^{3/2}$. Note that $\varepsilon' = \frac{3}{2}\varepsilon = \varepsilon(1 + \log \nu / \log k)$. Then we apply Lemma 1.3 with $f = f_2$ and $g = f_1$, where R_2 is the constant in (1.2.3) and R_0 is the constant in Lemma 1.2 for $f = f_2$. So, for $r \geq \max\{e, R_0, R_2\}$, we have

$$\begin{aligned} M(r^{k'}, f_1 \circ f_2) &\geq M(\nu r^k, f_1 \circ f_2) \\ &\geq M(M(r^k, f_2), f_1), \quad \text{by (1.2.3)} \\ &\geq M(M(r, f_2)^k, f_1), \quad \text{by (1.2.1)}. \end{aligned}$$

Hence, for $r \geq R' = \max\{e, R, R_0, R_2\}$,

$$\begin{aligned} (\log M(r^{k'}, f_1 \circ f_2))^\varepsilon &\geq (\log M(M(r, f_2)^k, f_1))^\varepsilon \\ &\geq k \log M(M(r, f_2), f_1), \quad \text{by (2.3.1)} \\ &\geq k \log M(r, f_1 \circ f_2). \end{aligned}$$

Hence, for $r \geq R'$,

$$\begin{aligned} (\log M(r^{k'}, f_1 \circ f_2))^{\varepsilon'} &= (\log M(r^{k'}, f_1 \circ f_2))^{(3/2)\varepsilon} \\ &\geq (k \log M(r, f_1 \circ f_2))^{3/2} \\ &= k' (\log M(r, f_1 \circ f_2))^{3/2} \\ &\geq k' \log M(r, f_1 \circ f_2), \end{aligned}$$

as required. So $f_1 \circ f_2$ is strongly log-regular. \square

2.4. Examples

In this section we construct two examples of functions with specific properties.

EXAMPLE 2.4.1. There exists a transcendental entire function of zero lower order and positive, finite order which is strongly log-regular.

EXAMPLE 2.4.2. There exists a transcendental entire function f which is log-regular such that $Q_2(f) \neq A(f)$. Hence, $Q(f) = A(f)$ but $Q_2(f) \neq Q(f)$.

In order to construct these functions we use Lemma 1.6.

We showed in Section 2.2 that all transcendental entire functions of finite order and positive lower order are strongly log-regular. However, a strongly log-regular function of finite order does not need to have positive lower order. Indeed, Example 2.4.1 gives a function of zero lower order and positive, finite order which is strongly log-regular.

PROOF OF EXAMPLE 2.4.1. We first take a fixed value of ε , say $\tilde{\varepsilon} \in (0, 1)$, and a fixed value of k , say \tilde{k} , such that $\tilde{k} > 2/\tilde{\varepsilon} \geq \frac{2\log(\tilde{k}+1)}{\log \tilde{k}}$ and construct a convex increasing function ϕ on \mathbb{R} such that:

- a) $\liminf_{t \rightarrow \infty} \frac{\log \phi(t)}{t} = 0$;
- b) $1 \leq \limsup_{t \rightarrow \infty} \frac{\log \phi(t)}{t} \leq \tilde{k}$;
- c) there exists $T > 0$ such that, for $t > T$,

$$(2.4.1) \quad \phi(\tilde{k}t) \geq (\tilde{k}\phi(t))^{1/\tilde{\varepsilon}}.$$

Once this is done, we show that this function ϕ satisfies (2.4.1) for *any* $\varepsilon \in (0, 1)$ with a suitable $k > 1$.

Take $\tilde{d} = 1/\tilde{\varepsilon}$. Then $\tilde{k} > \tilde{d} > 1$. Take $a_0 = 1$, and choose t_0 so large that

$$(2.4.2) \quad \frac{\log \tilde{k}}{t_0} < \frac{1}{2}$$

and

$$(2.4.3) \quad \tilde{k}^{\tilde{d}} e^{\tilde{d}t} \leq e^{\tilde{k}t}, \text{ for } t \geq t_0.$$

Then we set $t_n = \tilde{k}^n t_0$, for $n \in \mathbb{N}$, and define

$$(2.4.4) \quad a_n = e^{t_n}, \quad n = N_1, N_2, \dots, N_m, \dots,$$

where (N_m) is an increasing sequence, to be chosen shortly, and

$$(2.4.5) \quad a_n = (\tilde{k}a_{n-1})^{\tilde{d}}, \text{ elsewhere.}$$

We will show that, for each $m \in \mathbb{N}$, we can choose N_m so that

$$(2.4.6) \quad \frac{\log a_{N_m-1}}{t_{N_m-1}} < \frac{1}{2^m}$$

and

$$(2.4.7) \quad e^{t_{N_m}} \geq (\tilde{k}a_{N_m-1})^{\tilde{d}}.$$

Then we let ϕ be the real function that is linear on each of the intervals $[t_n, t_{n+1}]$ with $\phi(t_n) = a_n$, for $n \in \mathbb{N}$.

Suppose there is no $N_1 \in \mathbb{N}$ which satisfies (2.4.6) with $m = 1$. Then $a_n = (\tilde{k}a_{n-1})^{\tilde{d}}$ for all $n \in \mathbb{N}$. Hence,

$$(2.4.8) \quad \begin{aligned} \frac{\log a_n}{t_n} &= \frac{\log(\tilde{k}a_{n-1})^{\tilde{d}}}{t_n} \\ &= \frac{\tilde{d}}{\tilde{k}} \left(\frac{\log \tilde{k}}{t_{n-1}} + \frac{\log a_{n-1}}{t_{n-1}} \right). \end{aligned}$$

Now let $x_n = \frac{\log a_n}{t_n}$, $c = \tilde{d}/\tilde{k} < 1$ and $\varepsilon_{n-1} = \frac{\log \tilde{k}}{t_{n-1}}$. We have that

$$x_n = c(\varepsilon_{n-1} + x_{n-1}), \text{ for all } n \in \mathbb{N},$$

so

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n &\leq c \limsup_{n \rightarrow \infty} \varepsilon_{n-1} + c \limsup_{n \rightarrow \infty} x_{n-1} \\ &= c \limsup_{n \rightarrow \infty} x_{n-1}, \end{aligned}$$

as $\varepsilon_{n-1} \rightarrow 0$ as $n \rightarrow \infty$ and $c < 1$. Hence

$$\limsup_{n \rightarrow \infty} \frac{\log a_n}{t_n} = 0$$

and so we obtain a contradiction. Therefore, (2.4.6) is true for some $N_1 \in \mathbb{N}$.

Suppose now that (2.4.6) is true for $N_1, N_2, \dots, N_m \in \mathbb{N}$ but it fails to be true for all $n > N_m$. Following the above argument, we again obtain a contradiction and so there exists $N_{m+1} \in \mathbb{N}$ which satisfies (2.4.6).

It follows from (2.4.2), (2.4.6) and the fact that $\tilde{k} > 2\tilde{d} > 1$ that

$$\frac{\log(\tilde{k}a_{N_m-1})^{\tilde{d}}}{t_{N_m}} = \frac{\tilde{d}}{\tilde{k}} \left(\frac{\log \tilde{k}}{t_{N_m-1}} + \frac{\log a_{N_m-1}}{t_{N_m-1}} \right) \leq \frac{\log a_{N_m-1}}{t_{N_m-1}} < \frac{1}{2^m} < 1,$$

and so $e^{t_{N_m}} > (\tilde{k}a_{N_m-1})^{\tilde{d}}$, which means that (2.4.6) implies (2.4.7).

In order to prove a) we note that it follows from (2.4.6) that

$$\frac{\log \phi(t_{N_m-1})}{t_{N_m-1}} = \frac{\log a_{N_m-1}}{t_{N_m-1}} < \frac{1}{2^m}, \text{ for } m \in \mathbb{N},$$

and so

$$\liminf_{t \rightarrow \infty} \frac{\log \phi(t)}{t} \leq \liminf_{m \rightarrow \infty} \frac{1}{2^m} = 0.$$

We also note that it follows from (2.4.4) that

$$\frac{\log \phi(t_{N_m})}{t_{N_m}} = \frac{\log a_{N_m}}{t_{N_m}} = 1, \text{ for } m \in \mathbb{N},$$

and so, in order to prove b), it remains to show that

$$\limsup_{t \rightarrow \infty} \frac{\log \phi(t)}{t} \leq \tilde{k}.$$

It suffices to show that $\phi(t) \leq e^{\tilde{k}t}$ for large values of t . We will first show that $\phi(t_n) \leq e^{t_n}$, for n large enough.

Suppose that $\phi(t_n) \leq e^{t_n}$ for some n . Then either

$$\phi(t_{n+1}) = e^{t_{n+1}}$$

or

$$(2.4.9) \quad \phi(t_{n+1}) = (\tilde{k}\phi(t_n))^{\tilde{d}} \leq (\tilde{k}e^{t_n})^{\tilde{d}} \leq e^{\tilde{k}t_n} = e^{t_{n+1}},$$

by (2.4.3). In either case, we deduce that $\phi(t_n) \leq e^{t_n}$ implies that $\phi(t_{n+1}) \leq e^{t_{n+1}}$. Since $\phi(t_{N_m}) = e^{t_{N_m}}$, for all $m \in \mathbb{N}$, we conclude that $\phi(t_n) \leq e^{t_n}$, for $t_n \geq t_{N_1}$. Now take any $t \in [t_n, t_{n+1}]$, $n \geq N_1$. Then

$$\phi(t) \leq \phi(t_{n+1}) \leq e^{t_{n+1}} = e^{\tilde{k}t_n} \leq e^{\tilde{k}t},$$

and the result follows.

We now show that (2.4.1) is true for $t \geq t_0$. In order to do so, we consider the functions $g(t) = \phi(\tilde{k}t)$ and $h(t) = (\tilde{k}\phi(t))^{\tilde{d}}$. For each $n \geq 0$, g is a linear, increasing function

on $[t_n, t_{n+1}] = [t_n, \tilde{k}t_n]$ and h is convex on $[t_n, t_{n+1}]$. We will find the values of the two functions g and h at the endpoints of each interval and we will use the fact that the graph of a convex function which has the same or smaller values at the endpoints than a linear function is always below the graph of the linear function. Thus, to show that $h(t) = (\tilde{k}\phi(t))^{\tilde{d}} \leq \phi(\tilde{k}t) = g(t)$ for all $t \geq t_0$, it is sufficient to show that

$$(\tilde{k}\phi(t_n))^{\tilde{d}} \leq \phi(t_{n+1}), \text{ for } n \geq 0.$$

This is evidently true if (2.4.5) holds and if (2.4.4) holds it is true by (2.4.7).

Finally, we need to show that ϕ is convex. It suffices to show that the sequence of gradients $g_n = \frac{a_n - a_{n-1}}{t_n - t_{n-1}}$, $n \in \mathbb{N}$, of the line segments in the graph of ϕ is increasing, or equivalently that, for $n \in \mathbb{N}$,

$$(2.4.10) \quad \frac{a_{n+1} - a_n}{t_{n+1} - t_n} \geq \frac{a_n - a_{n-1}}{t_n - t_{n-1}}.$$

Since $t_n = \tilde{k}^n t_0$, for $n \in \mathbb{N}$, we need to show that

$$a_{n+1} \geq (\tilde{k} + 1)a_n - \tilde{k}a_{n-1}.$$

But

$$a_{n+1} + \tilde{k}a_{n-1} \geq (\tilde{k} + 1)a_n$$

since, by (2.4.7) and the fact that $\tilde{d} = \frac{1}{\tilde{\varepsilon}} \geq \frac{\log(\tilde{k}+1)}{\log \tilde{k}}$,

$$a_{n+1} \geq (\tilde{k}a_n)^{\tilde{d}} \geq \tilde{k}^{\tilde{d}}a_n \geq (\tilde{k} + 1)a_n, \text{ for } n \in \mathbb{N},$$

and the result follows.

We have constructed a function ϕ such that (2.4.1) holds for $\tilde{\varepsilon}$ which is a specific value of $\varepsilon \in (0, 1)$. In fact for any other $\varepsilon \in (0, \tilde{\varepsilon})$ we can find a large enough $k > 1$ such that (2.4.1) holds for the same function ϕ . Indeed, suppose first that (2.4.1) holds for $\varepsilon = \tilde{\varepsilon}$ and set $\tilde{d} = 1/\tilde{\varepsilon}$, as before.

Now take $\varepsilon \in (0, \tilde{\varepsilon})$ and suppose that $1/\varepsilon = d = \tilde{d}^p$, for some $n \leq p < n+1, n \in \mathbb{N}$. It follows from (2.4.1) that, for $t \geq t_0$,

$$\begin{aligned}
 \phi(\tilde{k}^{2n+2}t) &\geq (\tilde{k}\phi(\tilde{k}^{2n+1}t))^{\tilde{d}} = \tilde{k}^{\tilde{d}}\phi(\tilde{k}^{2n+1}t)^{\tilde{d}} \\
 &\geq \tilde{k}^{\tilde{d}}\tilde{k}^{\tilde{d}^2}\phi(\tilde{k}^{2n}t)^{\tilde{d}^2} \\
 (2.4.11) \quad &\geq \tilde{k}^{\tilde{d}+\tilde{d}^2+\dots+\tilde{d}^{2n+2}}\phi(t)^{\tilde{d}^{2n+2}}.
 \end{aligned}$$

We now show that

$$(2.4.12) \quad \tilde{k}^{\tilde{d}+\tilde{d}^2+\dots+\tilde{d}^{2n+2}}\phi(t)^{\tilde{d}^{2n+2}} \geq (\tilde{k}^{2n+2}\phi(t))^{\tilde{d}^p}.$$

As $p < n+1$, it suffices to show that

$$(2.4.13) \quad \tilde{d} + \tilde{d}^2 + \dots + \tilde{d}^{2n+2} \geq (2n+2)\tilde{d}^{n+1}.$$

We will prove (2.4.13) using the inequality of arithmetic and geometric means, which implies that

$$\begin{aligned}
 \frac{\tilde{d} + \tilde{d}^2 + \dots + \tilde{d}^{2n+2}}{2n+2} &\geq \sqrt[2n+2]{\tilde{d}\tilde{d}^2 \dots \tilde{d}^{2n+2}} \\
 &= \sqrt[2n+2]{\tilde{d}^{(2n+2)(2n+3)/2}} \\
 &= \tilde{d}^{(2n+3)/2} = \tilde{d}^{n+3/2} > \tilde{d}^{n+1},
 \end{aligned}$$

as required. Combining (2.4.11) and (2.4.12) gives

$$\phi(kt) \geq (k\phi(t))^\varepsilon, \text{ for } t \geq t_0,$$

where $k = \tilde{k}^{2n+2}$. Thus (2.4.1) holds for any $\varepsilon \in (0, \tilde{\varepsilon})$ and hence for any $\varepsilon \in (0, 1)$.

Now we can apply Lemma 1.6 to ϕ to give a transcendental entire function f such that

$$(2.4.14) \quad \log M(e^t, f) = \phi(t)(1 + \delta(t)),$$

where $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$. Then

$$\frac{\log \log M(e^t, f)}{t} = \frac{\log \phi(t)}{t} + \frac{O(\delta(t))}{t}$$

and so $\lambda(f) = 0$ and $1 \leq \rho(f) \leq \tilde{k}$, by properties (a) and (b) respectively.

It remains to show that f satisfies (2.1.3). We know that for any $\varepsilon \in (0, 1)$ there exists $k > d = 1/\varepsilon$ such that

$$(2.4.15) \quad \phi(kt) \geq (k\phi(t))^d, \text{ for } t \geq t_0.$$

Let $0 < \varepsilon' < \varepsilon$ and set $d' = 1/\varepsilon'$. Then, by (2.4.14) and (2.4.15),

$$\log M(e^{kt}, f) \geq \frac{1 + \delta(kt)}{(1 + \delta(t))^{d'}} (k \log M(e^t, f))^{d'}, \text{ for } t \geq t_0.$$

Since

$$\frac{1 + \delta(kt)}{(1 + \delta(t))^{d'}} \rightarrow 1 \text{ as } t \rightarrow \infty,$$

we deduce that

$$\log M(e^{kt}, f) \geq (k \log M(e^t, f))^{1/\varepsilon},$$

for large t and so f satisfies (2.1.3) for sufficiently large r with $r = e^t$. \square

Throughout the chapter, we are interested in sufficient conditions for $Q_2(f) = A(f)$. However, these two sets are not always equal. We now construct a function for which $Q_2(f)$ is not equal to $A(f)$.

PROOF OF EXAMPLE 2.4.2. We construct a transcendental entire function f which is log-regular and hence, by [71, Theorem 4.1], $Q(f) = A(f)$, but for which $Q_2(f) \neq A(f)$. Obviously, this function cannot be strongly log-regular. In order to construct such a function we will again use the result of Clunie (see Lemma 1.6). The idea is to find a real, increasing, convex function ϕ such that:

- there exist $k > 1$ and $d > 1$ such that

$$(2.4.16) \quad \phi(kt) \geq kd\phi(t), \text{ for large } t,$$

and

- if f is produced from ϕ using Lemma 1.6 then the iterates of the function $\mu_{2,\varepsilon}(r)$, for $\varepsilon \in (1/2, 1)$, grow much more slowly than the iterates of $M(r) = M(r, f)$.

Let $\phi(t) = t^2, t > 0$. Then ϕ is increasing and convex. Let $k > 1$ and $1 < d < k$. Then

$$\phi(kt) = k^2 t^2 > k d t^2 = k d \phi(t)$$

and so (2.4.16) is satisfied.

Now we apply Lemma 1.6 to ϕ to give a transcendental entire function f such that

$$(2.4.17) \quad \log M(e^t, f) = \phi(t)(1 + \delta(t)) = t^2(1 + \delta(t)),$$

where $\delta(t) \rightarrow 0$ as $t \rightarrow \infty$.

Now (2.4.16) implies that

$$\log M(e^{kt}, f) \geq k d \frac{1 + \delta(kt)}{1 + \delta(t)} \log M(e^t, f),$$

where

$$\frac{1 + \delta(kt)}{1 + \delta(t)} \rightarrow 1 \text{ as } t \rightarrow \infty.$$

Hence, there exists $1 < d' < d$ such that

$$\log M(e^{kt}, f) \geq k d' \log M(e^t, f),$$

for large t , and so, by (2.1.6), f is log-regular which implies that $Q(f) = A(f)$.

Now we show that, for $\varepsilon \in (1/2, 1)$, the iterates of $\mu_{2,\varepsilon}(r)$ grow more slowly than the iterates of $M(r)$.

By (2.4.17), we have

$$(2.4.18) \quad M(r) = \exp((\log r)^2(1 + \nu(r))),$$

where $\nu(r) = \delta(\log r) \rightarrow 0$, as $r \rightarrow \infty$, and

$$(2.4.19) \quad \mu_{2,\varepsilon}(r) = \exp((\log M(r))^\varepsilon) = \exp(((\log r)^2(1 + \nu(r)))^\varepsilon) = \exp((\log r)^{2\varepsilon}(1 + \nu(r))^\varepsilon).$$

Now fix $\varepsilon \in (1/2, 1)$. It then follows from (2.4.19) that there exists $R > 0$ such that we have $\mu_{2,\varepsilon}(r) > r$, for $r \geq R$ and so $\mu_{2,\varepsilon}^n(R) > R$, for $n \in \mathbb{N}$.

The idea is to show that, for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$ with $n > N$, we have

$$(2.4.20) \quad \mu_{2,\varepsilon}^{m+n}(R_0) < M^n(R_0),$$

for some $R_0 \geq R$. We then show that this implies that $Q_2(f) \neq A(f)$.

Since $\varepsilon \in (1/2, 1)$, it follows from (2.4.18) and (2.4.19) that there exist $R_1, R_2 > 0$ and $c, \tilde{c} \in \mathbb{R}$ such that

$$(2.4.21) \quad 1 < 2\varepsilon < \tilde{c} < c < 2,$$

$$(2.4.22) \quad M(r) \geq \exp((\log r)^c), \text{ for } r > R_1,$$

and

$$(2.4.23) \quad \mu_{2,\varepsilon}(r) \leq \exp((\log r)^{\tilde{c}}), \text{ for } r > R_2.$$

Hence, by (2.4.22) and (2.4.23), we obtain, for $n \in \mathbb{N}$,

$$M^n(r) \geq \exp((\log r)^{c^n}), \text{ for } r > R_1,$$

and

$$\mu_{2,\varepsilon}^n(r) \leq \exp((\log r)^{\tilde{c}^n}), \text{ for } r > R_2.$$

By (2.4.21), we can easily see that, for any $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$ with $n > N$, we have

$$\tilde{c}^{n+m} < c^n,$$

and hence

$$\mu_{2,\varepsilon}^{n+m}(r) \leq \exp((\log r)^{\tilde{c}^{n+m}}) < \exp((\log r)^{c^n}) \leq M^n(r), \text{ for } r > R_0,$$

where $R_0 = \max\{R, R_1, R_2\}$. Therefore, (2.4.20) is satisfied for $R_0 = \max\{R, R_1, R_2\}$. We will now show that (2.4.20) implies that $Q_2(f) \setminus A(f)$ is non-empty.

Now, by Theorem 1.4, with $a_n = \mu_{2,\varepsilon}^n(R)$, $n \in \mathbb{N}$, there exists a point ζ and a sequence (n_j) with $(n_j) \rightarrow \infty$ as $j \rightarrow \infty$, such that, for our function f ,

$$(2.4.24) \quad |f^n(\zeta)| \geq \mu_{2,\varepsilon}^n(R), \text{ for } n \in \mathbb{N},$$

and

$$(2.4.25) \quad |f^{n_j}(\zeta)| \leq M^2(\mu_{2,\varepsilon}^{n_j}(R)), \text{ for } j \in \mathbb{N}.$$

It follows from (2.4.24) that $\zeta \in Q_2(f)$. Also, (2.4.20) and (2.4.25) together imply that, for each $m \in \mathbb{N}$ and $n_j > m$, we have

$$\begin{aligned}
 |f^{(n_j-m+2)+m-2}(\zeta)| &= |f^{n_j}(\zeta)| \\
 &\leq M^2(\mu_{2,\varepsilon}^{n_j}(R)) \\
 &< M^2(M^{n_j-m}(R)) \\
 &= M^{n_j-m+2}(R).
 \end{aligned}$$

Hence, $\zeta \notin A(f)$, so $Q_2(f) \neq A(f)$, as required. □

CHAPTER 3

Generalising the quite fast escaping set

3.1. Introduction

In this chapter we look at the family of sets $Q_m(f)$ defined in §1.3.3 for the general case where $m \in \mathbb{N}$. Recall that

$$Q_m(f) = \{z : \exists \varepsilon \in (0, 1), \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq \mu_{m,\varepsilon}^n(R), \text{ for } n \in \mathbb{N}\},$$

where $\mu_{m,\varepsilon}, \varepsilon \in (0, 1)$, is defined by the equation

$$\log^m \mu_{m,\varepsilon}(r) = \varepsilon \log^m M(r), \quad m \in \mathbb{N},$$

whenever the right-hand side of the equality is well defined, and whenever there exists $R = R(f, \varepsilon) > 0$ such that $\mu_{m,\varepsilon}(r) > r$ for $r \geq R$. In Section 3.2 we give a large class of functions for which $\mu_{m,\varepsilon}(r)$ is greater than r for r large enough.

In Chapter 2 we considered the case $m = 2$, and we found regularity conditions that imply that $Q_2(f) = A(f)$. In particular, we proved that any transcendental entire function of finite order and positive lower order satisfies $Q_2(f) = A(f)$.

If $\mu_{m,\varepsilon}$ satisfies the generalised (2.1.4), that is, for any $\varepsilon \in (0, 1)$ and any $m \in \mathbb{N}$, there exist $R > 0$ and $k > 1$ such that

$$(3.1.1) \quad \mu_{m,\varepsilon}(r^k) \geq M(r)^k, \quad \text{for } r > R,$$

then we can show that $Q_m(f) = A(f)$. However, the larger m is, the more difficult it is for f to satisfy (3.1.1). For example, it is not hard to check that, for $m = 3$, $f(z) = e^z$ does not satisfy (3.1.1). In this chapter we give alternative, but more complicated, regularity conditions which, for any $m \geq 2$, guarantee that $Q_m(f) = A(f)$ for a wide range of functions f . We introduce new techniques that enable us to generalise our results from Chapter 2 to give the following result which holds for all $m \in \mathbb{N}$ and not just for $m = 2$.

THEOREM 3.1. *Let f be a transcendental entire function of finite order and positive lower order. Then $Q_m(f) = A(f)$, for $m \in \mathbb{N}$.*

Similarly to Corollary 2.2, Theorem 3.1 implies that functions in the class \mathcal{B} that have finite order satisfy $Q_m(f) = A(f)$.

We prove the theorem in two different ways. The first proof is based on a new regularity condition and the second on a growth condition. In Section 3.3 we give our first proof of Theorem 3.1 which is in two steps. We first introduce a new regularity condition which we call m -log-regularity and which implies that $Q_m(f) = A(f)$. Let f be a transcendental entire function. Then f is *m -log-regular* if and only if, for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that

$$(3.1.2) \quad \mu_{m,\varepsilon}(\exp^{m-1}(r^k)) \geq \exp^{m-1}(M(r)^k), \text{ for } r \geq R.$$

We are not aware whether there is any relationship between the different m -log regularity conditions. For $m = 1$ we obtain log-regularity (introduced in Section 1.2). We then show that any function that is m -log-regular satisfies $Q_m(f) = A(f)$. In the second step, we prove that all functions of finite order and positive lower order are m -log-regular.

In Section 3.4 we prove Theorem 3.1 in a different way, again in two steps. We give a growth condition that is sufficient for $Q_m(f) = A(f)$ and then we show that any transcendental entire function of finite order and positive lower order satisfies this growth condition.

In Chapter 2 we introduced a regularity condition called *strong log-regularity* which implies that $Q_2(f) = A(f)$. In Section 3.5 we show how strong log-regularity is related to 2-log-regularity. In particular, we prove that a strongly log-regular function of finite order is always 2-log regular and we give an example of a 2-log-regular function of finite order that fails to be strongly log-regular.

3.2. Properties of $Q_m(f)$

In this section we prove some basic properties of $Q_m(f)$. Just as for $\mu_{2,\varepsilon}$, in the general case we do not know a priori that, for any given transcendental entire function, $\mu_{m,\varepsilon}(r)$ is greater than r for r large enough. We show first that, for a large class of functions, there is

always a positive R such that $\mu_{m,\varepsilon}(r) > r$, for $r \geq R$, and hence for these functions $Q_m(f)$ is defined.

THEOREM 3.2. *Let f be a transcendental entire function, $m \geq 2$ and $\varepsilon \in (0, 1)$. If there exist $q > 0$, $r_0 > 0$ and $n \in \mathbb{N}$ such that*

$$(3.2.1) \quad M(r) \geq \exp^{n+1}((\log^n r)^q), \text{ for } r \geq r_0,$$

then, for any $c > 1$, there exists $R > 0$ such that

$$\mu_{m,\varepsilon}(r) > cr, \text{ for } r \geq R.$$

Note that (3.2.1) is true for all functions of positive lower order as well as some functions of zero lower order. In particular, it is true for all the functions in the class \mathcal{B} as they have lower order not less than $1/2$. In order to prove Theorem 3.2 we use the following inequality.

LEMMA 3.3. *For any $n \in \mathbb{N}$, $p \geq 1$ and $a_1, \dots, a_n, b_1, \dots, b_n > 0$ there exists $R > 0$ such that*

$$(3.2.2) \quad a_1 \log(a_2 \log \dots \log(a_n r) \dots) \geq \log(b_1 \log \dots \log((b_n r)^p) \dots), \text{ for } r \geq R.$$

PROOF. It suffices to prove (3.2.2) for $p > 1$. We use proof by induction. As $a_1 r \geq \log((b_1 r)^p)$, for r large enough, (3.2.2) is certainly true for $n = 1$. Suppose now that (3.2.2) is true for some $n \geq 2$. We will deduce that

$$(3.2.3) \quad a_1 \log(a_2 \log \dots \log(a_n \log(a_{n+1} r)) \dots) \geq \log(b_1 \log \dots \log(b_n \log((b_{n+1} r)^p) \dots)),$$

for r large enough. To do this, note first that, for r large enough,

$$a_1 \log(a_2 \log \dots \log(a_n \log(a_{n+1} r)) \dots) \geq \log(b_1 \log \dots \log((b_n \log(a_{n+1} r))^p) \dots),$$

by (3.2.2) for n . Then, in order to deduce (3.2.3) it suffices to show that

$$(3.2.4) \quad (b_n \log(a_{n+1} r))^p \geq b_n p \log(b_{n+1} r),$$

for r large enough. Note now that (3.2.4) is true since there exists $R(= R(n)) > 0$ such that

$$\frac{b_n^{p-1}}{p} (\log(a_{n+1} r))^p \geq \log(b_{n+1} r), \text{ for } r \geq R,$$

and the result follows. \square

PROOF OF THEOREM 3.2. By definition, $\mu_{m,\varepsilon}(r) = \exp^m(\varepsilon \log^m M(r))$, so we have to show that

$$(3.2.5) \quad \exp^m(\varepsilon \log^m M(r)) > cr, \text{ for } r \text{ large enough.}$$

We consider three different cases depending on the relative sizes of m and the positive integer n from (3.2.1).

a) Suppose that $n + 1 = m$. Then, by (3.2.1),

$$(3.2.6) \quad \begin{aligned} \exp^m(\varepsilon \log^m M(r)) &\geq \exp^m(\varepsilon \log^m(\exp^{n+1}((\log^n r)^q))) \\ &= \exp^m(\varepsilon (\log^n r)^q) \\ &> cr, \text{ for } r \text{ large enough,} \end{aligned}$$

since $\varepsilon (\log^n r)^q > \log \log^n cr = \log^m cr$, for r large enough.

b) Suppose that $n + 1 < m$. Then, by (3.2.1),

$$\exp^m(\varepsilon \log^m M(r)) \geq \exp^m(\varepsilon \log^m(\exp^{n+1}((\log^n r)^q))) = \exp^m(\varepsilon \log^{m-n-1}((\log^n r)^q)).$$

Hence, we need to show that, for any $c > 1$,

$$\exp^m(\varepsilon \log^{m-n-1}((\log^n r)^q)) > cr, \text{ for } r \text{ large enough,}$$

or, equivalently,

$$(3.2.7) \quad \varepsilon \log^{m-n-1}((\log^n r)^q) > \log^m cr, \text{ for } r \text{ large enough,}$$

which holds by applying Lemma 3.3 with n replaced by m , $p = 1$, $a_1 = \varepsilon$, $a_{m-n-1} = q$, $b_m = c$ and all the other coefficients equal to 1.

c) Finally, suppose that $n + 1 > m$. Then, by (3.2.1),

$$\exp^m(\varepsilon \log^m M(r)) \geq \exp^m(\varepsilon \log^m(\exp^{n+1}((\log^n r)^q))) = \exp^m(\varepsilon \exp^{n+1-m}((\log^n r)^q)).$$

Hence, we need to show that, for any $c > 1$,

$$\exp^m(\varepsilon \exp^{n+1-m}((\log^n r)^q)) > cr, \text{ for } r \text{ large enough,}$$

or, equivalently,

$$(3.2.8) \quad \varepsilon \exp^{n+1-m}((\log^n r)^q) > \log^m cr, \text{ for } r \text{ large enough.}$$

Note now that (3.2.8) is equivalent to

$$(3.2.9) \quad (\log^n r)^q > \log^{n+1-m} \left(\frac{1}{\varepsilon} \log^m cr \right), \text{ for } r \text{ large enough.}$$

If we apply Lemma 3.3 with n replaced by $n+2$, $p=1$, $a_1=q$, $b_{n+2-m}=1/\varepsilon$, $b_{n+2}=c$ and the rest of the coefficients equal to 1 we obtain

$$q \log^{n+1} r > \log^{n+2-m} \left(\frac{1}{\varepsilon} \log^m cr \right),$$

for r large enough and so (3.2.9) follows. \square

We now show that $Q_m(f)$ has some basic properties similar to those of $I(f)$, $A(f)$ and $Q(f)$. In order to prove one of the properties we need the following lemma (see [11, Lemma 7]).

LEMMA 3.4. *Let f be a transcendental entire function, $U \subset I(f)$ be a simply connected Fatou component and let K be a compact subset of U . Then there exists $C > 1$ and $N \in \mathbb{N}$ such that*

$$|f^n(z_0)| \leq C |f^n(z_1)|, \text{ for } z_0, z_1 \in K, n \geq N.$$

We have the following result.

THEOREM 3.5. *Let f be a transcendental entire function and $m \in \mathbb{N}$ such that $Q_m(f)$ is defined. Then*

$$Q_m(f) \neq \emptyset, \quad Q_m(f) \cap J(f) \neq \emptyset, \quad \text{and} \quad J(f) = \overline{Q_m(f) \cap J(f)}.$$

If, in addition, for any $c > 1$, there exist $R > 0$ and $\varepsilon \in (0, 1)$ such that

$$(3.2.10) \quad \mu_{m,\varepsilon}(r) > cr, \quad \text{for } r \geq R,$$

then

$$J(f) = \partial Q_m(f),$$

and $\overline{Q_m(f)}$ has no bounded components.

PROOF. All the properties above hold for $A(f)$ (see [68]). As $A(f) \subset Q_m(f)$, we certainly have $Q_m(f) \neq \emptyset$ and $Q_m(f) \cap J(f) \neq \emptyset$. Also $J(f) = \overline{A(f) \cap J(f)} \subset \overline{Q_m(f) \cap J(f)}$.

Since $J(f)$ is closed, we also have $\overline{Q_m(f) \cap J(f)} \subset J(f)$ and so the third property is also true.

In order to prove the two remaining properties, we follow the arguments in the proof of [71, Theorem 2.1]. Note first that $Q_m(f)$ is infinite and completely invariant under f which, since $J(f)$ is the smallest closed completely invariant set with at least three points, implies that $J(f) \subset \overline{Q_m(f)}$. But any open subset of $Q_m(f)$ is contained in $F(f)$ since it contains no periodic points of f , and so $J(f) \subset \partial Q_m(f)$.

Suppose now that $\partial Q_m(f) \cap U \neq \emptyset$, where U is a Fatou component. Then $Q_m(f) \cap U \neq \emptyset$, and we take $z \in Q_m(f) \cap U$. Then there will be a disc Δ such that $z \in \Delta$ and $\overline{\Delta} \subset U$. If U is simply connected then, by applying Lemma 3.4, we have that there exists $C > 1$ and $N \in \mathbb{N}$ such that

$$|f^n(z')| \geq C|f^n(z)|,$$

for any $z' \in \Delta$ and $n \geq N$. If (3.2.10) holds there exist $R > 0$ and $\varepsilon \in (0, 1)$ such that $\mu_{m,\varepsilon}(r) > r$, $r \geq R$ and $\ell \in \mathbb{N}$ such that, for $m \in \mathbb{N}$,

$$|f^{N+m+\ell}(z')| \geq C|f^{N+m+\ell}(z)| \geq C\mu_{m,\varepsilon}^{N+m}(R),$$

and so

$$|f^{n+\ell}(z')| \geq \mu_{m,\varepsilon}^{n-1}(R),$$

for all $n \in \mathbb{N}$, by (3.2.10). Therefore, any point in the neighbourhood Δ of z lies in $Q_m(f)$ which gives a contradiction. If the Fatou component U is multiply connected then $U \subset A(f) \subset Q_m(f)$ (see [70, Theorem 2]) and so again there is a contradiction. Hence, $\partial Q_m(f) \subset J(f)$.

Finally, if $\overline{Q_m(f)}$ has a bounded component, E say, then there is an open topological annulus A lying in the complement of $\overline{Q_m(f)}$ that surrounds E . Since $\overline{Q_m(f)}$ is completely invariant under f , A is contained in $F(f)$ by Montel's theorem. But from the previous property, $J(f) = \partial Q_m(f)$ and so a is contained in a multiply connected Fatou component. As any multiply connected Fatou component is contained in $A(f) \subset Q_m(f)$ we deduce that $A \subset Q_m(f)$ which gives a contradiction. \square

3.3. Regularity conditions for $Q_m(f) = A(f)$

In this section, we use regularity conditions to prove Theorem 3.1. In Section 3.1 we defined m -log-regularity, which is a sufficient condition for $Q_m(f)$ to be equal to $A(f)$. In fact, there also exists another regularity condition called m -weak-regularity that is equivalent to $Q_m(f) = A(f)$. We will show later that m -log-regularity is stronger than m -weak-regularity and hence if f is m -log-regular then $Q_m(f) = A(f)$. Finally, we will use these ideas in order to prove Theorem 3.1. Note that m -log-regularity is easier to check than m -weak-regularity which is defined as follows:

Let $R > 0$ be any value such that $M(r) > r$ for $r \geq R$. We say that f is *m -weakly regular* if for any $\varepsilon \in (0, 1)$ there exists $r = r(R) > 0$ such that

$$\mu_{m,\varepsilon}^n(r) \geq M^n(R), \text{ for } n \in \mathbb{N},$$

or, equivalently, if there exists $\ell = \ell(R) \in \mathbb{N}$ such that

$$\mu_{m,\varepsilon}^{n+\ell}(R) \geq M^n(R), \text{ for } n \in \mathbb{N}.$$

For $m = 1$ we have the weak-regularity that was introduced by Rippon and Stallard in [71].

We will show that m -weak-regularity is a necessary and sufficient condition for f to satisfy $Q_m(f) = A(f)$.

THEOREM 3.6. *Let f be a transcendental entire function. Then f is m -weakly regular if and only if $Q_m(f)$ is defined and $Q_m(f) = A(f)$.*

PROOF. Suppose that f is m -weakly regular and let $R > 0$ be such that $M(r) > r$ for $r \geq R$. Then there exists $r = r(R) > 0$ such that

$$\mu_{m,\varepsilon}^n(r) \geq M^n(R), \text{ for } n \in \mathbb{N},$$

and so $Q_m(f)$ is defined.

If $z \in Q_m(f)$, then there exist $\varepsilon \in (0, 1)$ and $\ell \in \mathbb{N}$ such that

$$|f^{n+\ell}(z)| \geq \mu_{m,\varepsilon}^n(R), \text{ for } n \in \mathbb{N}.$$

Let $r = r(R)$ be as above. Then there exists $N \in \mathbb{N}$ such that $\mu_{m,\varepsilon}^N(R) > r$ so

$$|f^{n+\ell+N}(z)| \geq \mu_{m,\varepsilon}^{n+N}(R) \geq \mu_{m,\varepsilon}^n(r) \geq M^n(R), \text{ for } n \in \mathbb{N},$$

and hence $z \in A(f)$. Thus $Q_m(f) \subset A(f)$. Clearly $A(f) \subset Q_m(f)$ and so we have $Q_m(f) = A(f)$ as claimed.

In order to show that the opposite direction of the theorem is also true we will prove that if f is not m -weakly regular and $Q_m(f)$ is defined then $Q_m(f) \setminus A(f)$ is non-empty. Take $R > 0$ such that $\mu_{m,\varepsilon}(r) > r$, for $r \geq R$. Since f is not weakly-log-regular, for any $\ell \in \mathbb{N}$ there exists $n(\ell) \in \mathbb{N}$ such that $\mu_{m,\varepsilon}^{n(\ell)+\ell}(R) < M^{n(\ell)}(R)$ and hence, for any $n \in \mathbb{N}$ with $n > n(\ell)$, we have

$$(3.3.1) \quad \mu_{m,\varepsilon}^{n+\ell}(R) < M^n(R).$$

Now, by Theorem 1.4, with $a_n = \mu_{m,\varepsilon}^n(R)$, $n \in \mathbb{N}$, there exists a point ζ and a sequence $n_j \rightarrow \infty$ as $j \rightarrow \infty$, such that

$$(3.3.2) \quad |f^{n_j}(\zeta)| \geq \mu_{m,\varepsilon}^{n_j}(R), \text{ for } n_j \in \mathbb{N},$$

and

$$(3.3.3) \quad |f^{n_j}(\zeta)| \leq M^2(\mu_{m,\varepsilon}^{n_j}(R)), \text{ for } j \in \mathbb{N}.$$

It follows from (3.3.2) that $\zeta \in Q_m(f)$. Also, (3.3.1) and (3.3.3) together imply that, for each $\ell \in \mathbb{N}$ and sufficiently large values of j , we have

$$\begin{aligned} |f^{(n_j-\ell+2)+\ell-2}(\zeta)| &= |f^{n_j}(\zeta)| \\ &\leq M^2(\mu_{m,\varepsilon}^{n_j}(R)) \\ &< M^2(M^{n_j-\ell}(R)) \\ &= M^{n_j-\ell+2}(R). \end{aligned}$$

Hence, $\zeta \notin A(f)$, so $Q_m(f) \neq A(f)$, as required. \square

We now give the proof of Theorem 3.1. The proof is in two steps. First, we prove the following result which implies that all m -log-regular functions satisfy $Q_m(f) = A(f)$.

THEOREM 3.7. *Let f be a transcendental entire function. If f is m -log-regular, then f is m -weakly regular and hence $Q_m(f) = A(f)$.*

PROOF. Suppose that f is m -log-regular and let $0 < \varepsilon < 1$. Let $R > 0$ be so large that $M(r) > r$ for $r \geq R$. Since f is m -log-regular, for any $\varepsilon \in (0, 1)$ there exists $r_0 \geq R$ and $k > 1$ such that

$$\mu_{m,\varepsilon}(\exp^{m-1}(r^k)) \geq \exp^{m-1}(M(r)^k), \text{ for } r \geq r_0.$$

Hence,

$$\begin{aligned} \mu_{m,\varepsilon}(\mu_{m,\varepsilon}(\exp^{m-1}(r^k))) &\geq \mu_{m,\varepsilon}(\exp^{m-1}(M(r)^k)) \\ &\geq \exp^{m-1}((M(M(r)))^k) \end{aligned}$$

and so, using this argument repeatedly, we have

$$\mu_{m,\varepsilon}^n(\exp^{m-1}(r^k)) \geq \exp^{m-1}(M^n(r)^k), \text{ for } r \geq r_0 \text{ and } n \in \mathbb{N}.$$

Thus, whenever $r \geq r_0$, we have

$$\mu_{m,\varepsilon}^n(\exp^{m-1}(r^k)) \geq M^n(r) \geq M^n(R), \text{ for } n \in \mathbb{N},$$

and so f is m -weakly regular. Hence, by Theorem 3.6, $Q_m(f) = A(f)$. \square

The second part of the proof of Theorem 3.1 is to show that all functions of finite order and positive lower order are m -log-regular. In order to prove this we will need the following lemma.

LEMMA 3.8. *For any $n \in \mathbb{N}$ and any $d > 0$, $q, q' \in (0, 1)$, there exists $R > 0$ such that*

$$(3.3.4) \quad \log^n(r^q) > d(\log^n r)^{q'}, \text{ for } r \geq R.$$

PROOF. We will prove (3.3.4) using induction. For $n = 1$,

$$q \log r > d(\log r)^{q'}, \text{ for } r \text{ large enough.}$$

Suppose that (3.3.4) is true for some $n \in \mathbb{N}$. Then

$$d(\log^{n+1} r)^{q'} = d(\log^n(\log r))^{q'} < \log^n((\log r)^q), \text{ for } r \text{ large enough.}$$

Hence, in order to prove (3.3.4) it suffices to show that there exists $R > 0$ such that

$$\log^{n+1}(r^q) > \log^n((\log r)^q), \text{ for } r \geq R, n \in \mathbb{N},$$

or equivalently that

$$\log(r^q) > (\log r)^q, \text{ for } r \geq R,$$

which is true, and so, the result follows. \square

We now prove the following result.

THEOREM 3.9. *Let f be a transcendental entire function of finite order and positive lower order. Then f is m -log-regular.*

PROOF. Let f be a transcendental entire function of finite order and positive lower order. We begin by noting that there exist $0 < q < p$ such that

$$(3.3.5) \quad e^{r^q} \leq M(r) \leq e^{r^p}, \text{ for } r \text{ large enough}$$

By the definition of $\mu_{m,\varepsilon}$, in order to prove that f is m -log-regular we need to show that for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that f satisfies (3.1.2) or, equivalently,

$$(3.3.6) \quad \varepsilon \log^m M(\exp^{m-1}(r^k)) \geq \log^m(\exp^{m-1}(M(r)^k)), \text{ for } r \text{ large enough.}$$

But (3.3.5) implies that

$$\log^m M(\exp^{m-1}(r^k)) \geq \log^{m-1}((\exp^{m-1}(r^k))^q)$$

and

$$\log^m(\exp^{m-1}(M(r)^k)) \leq \log^m(\exp^m(kr^p)) = kr^p,$$

and so (3.3.6) is implied by

$$\varepsilon \log^{m-1}((\exp^{m-1}(r^k))^q) \geq kr^p,$$

that is,

$$(3.3.7) \quad (\exp^{m-1}(r^k))^q \geq \exp^{m-1}\left(\frac{kr^p}{\varepsilon}\right), \text{ for } r \text{ large enough.}$$

We set $r^k = \log^{m-1} s$ and (3.3.7) becomes

$$(3.3.8) \quad \log^{m-1}(s^q) \geq \frac{k}{\varepsilon}(\log^{m-1} s)^{p/k}, \text{ for } s \text{ large enough.}$$

For $m = 1$, if we choose $k > p/q$ then, for any $\varepsilon \in (0, 1)$, (3.3.8) holds for s large enough. For $m > 1$, if we choose $k > p$ then, for any $\varepsilon \in (0, 1)$, (3.3.8) holds for s large enough, by Lemma 3.8. \square

It is easy to see that if we combine Theorem 3.7 and Theorem 3.9 we obtain Theorem 3.1.

3.4. Growth conditions for $Q_m(f) = A(f)$

In this section we give a second proof of Theorem 3.1 by introducing a growth condition, given in the following theorem, that implies that $Q_m(f) = A(f)$. The same lower bound for $m = 2$ appears in [69, Theorem 6].

THEOREM 3.10. *Let f be a transcendental entire function, $m \geq 2$ and $\phi_m(t) = \log^{m-1} M(\exp^{m-1}(t))$, where $M(\exp^{m-1}(t)) = M(\exp^{m-1}(t), f)$. If there exist $0 < q < 1$ and $0 < \tilde{q} < \infty$ such that, for some $n \geq 0$,*

(3.4.1)

$$\exp^{n+m-1}((\log^{n+m-2} t)^q) \leq \phi_m(t) \leq \exp^{n+m-1}((\log^{n+m-2} t)^{\tilde{q}}), \text{ for } t \text{ large enough,}$$

then

(i) *for any $d > 1$ there exists $t_0 > 0$ such that*

$$\phi_m(\psi_m(t)) \geq (\psi_m(\phi_m(t)))^d, \text{ for } t \geq t_0,$$

$$\text{where } \psi_m(t) = \exp^{n+m-1}((\log^{n+m-1} t)^p), \text{ } pq > 1;$$

(ii) *f is m -weakly regular and so $Q_m(f) = A(f)$.*

REMARK 3.4.1. As $0 < q < 1$, the left bound in (3.4.1) becomes smaller as n increases. If we also take $\tilde{q} > 1$, then the right bound increases with n and hence the condition (3.4.1) is more easily satisfied for larger n . As we will prove in Theorem 3.11, all functions of positive lower order and finite order satisfy (3.4.1).

PROOF. (i) We have that

$$\begin{aligned} \psi_m(\phi_m(t)) &\leq \exp^{n+m-1}((\log^{n+m-1}(\exp^{n+m-1}(\log^{n+m-2} t)^{\tilde{q}}))^p) \\ &= \exp^{n+m-1}((\log^{n+m-2} t)^{\tilde{q}p}) \end{aligned}$$

and also

$$\begin{aligned}
\phi_m(\psi_m(t)) &\geq \exp^{n+m-1}((\log^{n+m-2}(\exp^{n+m-1}((\log^{n+m-1} t)^p)))^q) \\
&= \exp^{n+m-1}((\exp((\log^{n+m-1} t)^p))^q) \\
&= \exp^{n+m}(q(\log^{n+m-1} t)^p) \\
&\geq \exp^{n+m}((\log^{n+m-1} t)^{pq}) \\
&\geq (\exp^{n+m-1}((\log^{n+m-2} t)^{p\tilde{q}}))^d, \text{ for any } d > 1 \text{ and for } t \text{ large enough,}
\end{aligned}$$

since putting $w = \log^{n+m-1} t$ gives

$$\begin{aligned}
\frac{\exp^{n+m-1}((\log^{n+m-1} t)^{pq})}{\exp^{n+m-2}((\log^{n+m-2} t)^{p\tilde{q}})} &= \frac{\exp^{n+m-1}(w^{pq})}{\exp^{n+m-2}((e^w)^{p\tilde{q}})} \\
&= \frac{\exp^{n+m-1}(w^{pq})}{\exp^{n+m-2}(e^{p\tilde{q}w})} \\
&= \frac{\exp^{n+m-1}(w^{pq})}{\exp^{n+m-1}(p\tilde{q}w)} \rightarrow \infty \text{ as } w \rightarrow \infty
\end{aligned}$$

and so

$$\exp^{n+m-1}((\log^{n+m-1} t)^{pq}) \geq d \exp^{n+m-2}((\log^{n+m-2} t)^{p\tilde{q}}), \text{ for } t \text{ large enough.}$$

Thus

$$\phi_m(\psi_m(t)) \geq (\psi_m(\phi_m(t)))^d, \quad d > 1, \text{ for } t \text{ large enough.}$$

(ii) Now let $\phi_{m,\varepsilon}(t) = \phi_m(t)^\varepsilon$ and note that from the definition of ϕ_m ,

$$M^n(r) = \exp^{m-1}(\phi_m^n(\log^{m-1} r)).$$

Note also that

$$\begin{aligned}
\mu_{m,\varepsilon}(r) &= \exp^m(\varepsilon \log^m M(r)) \\
&= \exp^m(\varepsilon \log^m(\exp^{m-1} \phi_m(\log^{m-1} r))) \\
&= \exp^m(\log(\phi_{m,\varepsilon}(\log^{m-1} r))) \\
&= \exp^{m-1}(\phi_{m,\varepsilon}(\log^{m-1} r)),
\end{aligned}$$

and so,

$$\mu_{m,\varepsilon}^n(r) = \exp^{m-1}(\phi_{m,\varepsilon}^n(\log^{m-1} r)).$$

Hence, in order to show that there exists $r = r(R) > 0$ such that

$$\mu_{m,\varepsilon}^n(r) \geq M^n(R), \text{ for } n \in \mathbb{N},$$

it suffices to show that there exists $r = r(R) > 0$ such that

$$\phi_{m,\varepsilon}^n(r) \geq \phi_m^n(R), \text{ for } n \in \mathbb{N}.$$

We showed in (i) that, for any $d > 1$, $\phi_m(\psi_m(t)) \geq (\psi_m(\phi_m(t)))^d$, if t is sufficiently large, or, equivalently, that given $\varepsilon > 0$

$$\phi_{m,\varepsilon}(\psi_m(t)) \geq \psi_m(\phi_m(t)), \text{ for } t \text{ large enough.}$$

Therefore,

$$\phi_{m,\varepsilon}(s) \geq \psi_m(\phi_m(\psi_m^{-1}(s))), \text{ for } s \text{ large enough.}$$

Since $\psi_m(t) \geq t$, by iterating we obtain

$$\phi_{m,\varepsilon}^n(s) \geq \psi_m(\phi_m^n(\psi_m^{-1}(s))) \geq \phi_m^n(\psi_m^{-1}(s)), \text{ for } s \text{ large enough.}$$

The result follows. \square

In order to complete the proof of Theorem 3.1 it remains to show that Theorem 3.10 can be applied to functions of finite order and positive lower order.

THEOREM 3.11. *Let f be a transcendental entire function of finite order and positive lower order. Then f satisfies the hypotheses of Theorem 3.10 and hence $Q_m(f) = A(f)$.*

PROOF. As f is of finite order and positive lower order, (3.3.5) implies that, for $m \geq 2$ there exist $q \in (0, 1)$ and $p \in (q, \infty)$ such that

$$(3.4.2) \quad \log^{m-2}((\exp^{m-1} t)^q) \leq \phi_m(t) = \log^{m-1} M(\exp^{m-1} t) \leq \log^{m-2}((\exp^{m-1} t)^p),$$

for t large enough.

In order to show that (3.4.2) implies (3.4.1), it suffices to show that

$$(3.4.3) \quad \exp^{m-1}((\log^{m-2} t)^q) \leq \log^{m-2}((\exp^{m-1} t)^q)$$

and

$$(3.4.4) \quad \log^{m-2}((\exp^{m-1} t)^p) \leq \exp^{m-1}((\log^{m-2} t)^p),$$

for t large enough. Note that (3.4.3) is equivalent to

$$(\log^{m-2} t)^q \leq \log^{2m-3}(\exp^{m-1} t)^q, \text{ for } t \text{ large enough,}$$

which, for $s = \exp^{m-1} t$, becomes

$$(3.4.5) \quad (\log^{2m-3} s)^q \leq \log^{2m-3} s^q, \text{ for } s \text{ large enough.}$$

By Lemma 3.8, (3.4.5) holds for s large enough and hence so does (3.4.3).

Similarly, using Lemma 3.8, one can show that (3.4.4) is true.

Therefore, the hypotheses of Theorem 3.10 are satisfied for q and $p = \tilde{q}$. \square

3.5. 2-log-regularity and strong log-regularity

In Chapter 2 we introduced a sufficient condition for $Q_2(f) = A(f)$ called strong log-regularity. Recall that a transcendental entire function f is strongly log-regular if, for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that, for $r > R$,

$$\log M(r^k) \geq (k \log M(r))^{1/\varepsilon}.$$

Note that strong log-regularity implies log-regularity (see Section 2.1), which, as mentioned in Section 3.1, is the same as 1-log-regularity.

Both strong log-regularity and 2-log-regularity imply $Q_2(f) = A(f)$ and also any transcendental entire function of finite order and positive lower order is both strongly log-regular and 2-log-regular. Therefore it is of interest to know how these two conditions are related. For a function of finite order we have the following result.

THEOREM 3.12. *Let f be a transcendental entire function of finite order. If f is strongly log-regular then f is 2-log-regular.*

PROOF. As f is of finite order, (3.3.5) implies that there exists $p \geq 0$, such that

$$(3.5.1) \quad \log M(r) \leq r^p, \text{ for } r \text{ large enough.}$$

Also since f is strongly log-regular, for any $\varepsilon \in (0, 1)$, there exist $R > 0$ and $k > 1$ such that

$$(3.5.2) \quad \log M(r^k) \geq (k \log M(r))^{1/\varepsilon}, \text{ for } r > R.$$

In order to show that f is 2-log-regular we will show that, for any $\varepsilon \in (0, 1)$,

$$\mu_{2,\varepsilon}(\exp(r^k)) \geq \exp(M(r)^k), \text{ for } r \text{ large enough;}$$

that is, using the definition of $\mu_{2,\varepsilon}(r)$,

$$(3.5.3) \quad (\log M(\exp(r^k)))^\varepsilon \geq M(r)^k, \text{ for } r \text{ large enough.}$$

It is obvious from the definition of 2-log-regularity that if the condition holds for any $\varepsilon \in (0, 1/e^p)$ it will hold for any $\varepsilon \in (0, 1)$ and so we now fix $\varepsilon \in (0, 1/e^p)$ and show that (3.5.3) holds for this value of ε .

Consider now

$$(3.5.4) \quad n = \left\lfloor \frac{k \log r - \log \log r}{\log k} \right\rfloor,$$

where $[x]$ denotes the integer part of the real number x . Then $k^n \leq r^k / \log r$, which gives us that $\exp(r^k) \geq r^{k^n}$. Hence

$$(\log M(\exp(r^k)))^\varepsilon \geq (\log M(r^{k^n}))^\varepsilon,$$

and by applying (3.5.2) n times, we deduce that

$$(\log M(\exp(r^k)))^\varepsilon \geq k^{1+1/\varepsilon+\dots+1/\varepsilon^{n-1}} (\log M(r))^{1/\varepsilon^{n-1}}, \text{ for } r \text{ large enough.}$$

Therefore, it suffices to show that

$$(\log M(r))^{1/\varepsilon^{n-1}} \geq M(r)^k,$$

or, equivalently, that

$$\left(\frac{1}{\varepsilon}\right)^{n-1} \log \log M(r) \geq k \log M(r), \text{ for } r \text{ large enough.}$$

By (3.5.1) it is sufficient to show that

$$\left(\frac{1}{\varepsilon}\right)^{n-1} \geq kr^p,$$

or, equivalently,

$$(3.5.5) \quad (n-1) \log \frac{1}{\varepsilon} \geq \log k + p \log r, \text{ for } r \text{ large enough.}$$

In order to show that (3.5.5) is true we first note that it follows from (3.5.4) that

$$n - 1 \geq \frac{k \log r}{\log k} - \frac{\log \log r + 2 \log k}{\log k},$$

and so

$$(3.5.6) \quad (n - 1) \log \frac{1}{\varepsilon} - p \log r \geq \left(\left(\log \frac{1}{\varepsilon} \right) \frac{k}{\log k} - p \right) \log r - \left(\log \frac{1}{\varepsilon} \right) \frac{\log \log r + 2 \log k}{\log k}.$$

Since $\log \frac{1}{\varepsilon} > p$, there exists $R_0 > 0$ such that

$$\left(\left(\log \frac{1}{\varepsilon} \right) \frac{k}{\log k} - p \right) \log r \geq \left(\log \frac{1}{\varepsilon} \right) \frac{\log \log r + 2 \log k}{\log k} + \log k, \text{ for } r \geq R_0.$$

Together with (3.5.6), this is sufficient to prove (3.5.5). \square

The converse of Theorem 3.12 is not always true though. We now use a function, that was constructed by Rippon and Stallard in [71, Example 6.1], in order to prove that there exists a 2-log-regular function of finite order which is not strongly log-regular.

We will need the following result:

LEMMA 3.13. *Let ϕ and ψ be real functions defined on $(0, \infty)$ with $\liminf_{t \rightarrow \infty} \phi(t) > 1$, $\liminf_{t \rightarrow \infty} \psi(t) > 1$ and such that*

$$\phi(t) \sim \psi(t), \text{ as } t \rightarrow \infty.$$

Then, for any $\varepsilon \in (0, 1)$, there exist $t_0 > 0$ and $k > 1$ such that

$$(3.5.7) \quad \phi(e^{kt}) \geq \exp\left(\frac{k}{\varepsilon} \phi(t)\right), \text{ for } t \geq t_0$$

if and only if there exist $t_1 > 0$ and $k' > 1$ such that

$$\psi(e^{k't}) \geq \exp\left(\frac{k'}{\varepsilon} \psi(t)\right), \text{ for } t \geq t_1.$$

PROOF. Let

$$(3.5.8) \quad \phi(t) = \psi(t)(1 + \epsilon(t)),$$

where $\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$ and suppose that ϕ satisfies (3.5.7). Then

$$(3.5.9) \quad \log \phi(t^k) \geq \frac{k}{\varepsilon} \phi(\log t), \text{ for } \log t \geq t_0.$$

It follows from (3.5.8) and (3.5.9) that

$$\log \psi(t^k) + \log(1 + \epsilon(t^k)) \geq \frac{k}{\epsilon} \psi(\log t)(1 + \epsilon(\log t)), \text{ for } \log t \geq t_0.$$

Hence, since $1 + \epsilon(\log t) \rightarrow 1$ as $t \rightarrow \infty$, $\log(1 + \epsilon(t^k)) \rightarrow 0$, as $t \rightarrow \infty$, and $\liminf_{t \rightarrow \infty} \phi(t) > 1$, $\liminf_{t \rightarrow \infty} \psi(t) > 1$, there exist $k' > 1$ and $t_1 > 0$, such that

$$\log(\psi(t^{k'})) \geq \frac{k'}{\epsilon} \psi(\log t), \text{ for } t \geq t_1,$$

as claimed. \square

EXAMPLE 3.5.1. There exists a transcendental entire function of finite order that is 2-log-regular but not strongly log-regular.

PROOF. The main idea of the proof is to use a function f constructed by Rippon and Stallard [71, Example 6.1], which has order zero and is not log-regular (and hence is not strongly log-regular) and show that f is 2-log-regular.

In order to show that f is 2-log-regular we need to show that, for any $\epsilon > 0$, there exist $r_0 > 0$ and $k > 1$ such that

$$\mu_{2,\epsilon}(\exp(r^k)) \geq \exp(M(r)^k), \text{ for } r \geq r_0,$$

or equivalently,

$$\log(M(\exp(r^k))) \geq M(r)^{k/\epsilon}, \text{ for } r \geq r_0.$$

Hence, the condition we have to prove for $\psi(t) = \log M(e^t)$ is that, for any $\epsilon \in (0, 1)$, there exist $t_0 > 0$ and $k > 1$ such that

$$(3.5.10) \quad \psi(\exp(kt)) \geq \exp\left(\frac{k}{\epsilon} \psi(t)\right), \text{ for } t \geq t_0.$$

The function f in [71, Example 6.1] was found by first constructing a real, increasing, convex function ϕ with certain properties and then using the result of Clunie that we stated in Section 1.2 (Lemma 1.6) in order to obtain a transcendental entire function f such that $\psi(t) = \log M(e^t) \sim \phi(t)$.

Let $\mu(t) = \exp(t^{1/2})$, $t \geq 0$. Then take $t_0 > 1$ so large that $\exp(\frac{3}{4}t^{1/2}) > t$ for $t \geq t_0$, and define $t_n = \mu^n(t_0)$ and $k_n = t_{n+1}^{1/4}$, $n \geq 0$.

Rippon and Stallard defined ϕ as follows:

$$\phi(t) = \begin{cases} \mu_n(t), & t \in [t_{n+1}/k_n, t_{n+1}], \\ \mu(t), & \text{otherwise,} \end{cases}$$

where $\mu_n(t)$ denotes the linear function such that $\mu_n(t) = \mu(t)$ for $t = t_{n+1}/k_n = t_{n+1}^{3/4}$, $t = t_{n+1}$.

We will first show that for any $\varepsilon \in (0, 1)$, there exist $t' > 0$ and $k' > 1$ such that

$$(3.5.11) \quad \phi(\exp(k't)) \geq \exp\left(\frac{k'}{\varepsilon}\phi(t)\right), \quad \text{for } t \geq t'.$$

Let $\varepsilon \in (0, 1)$. When $\phi(t) = \mu(t) = \exp(t^{1/2})$, we have

$$\phi(\exp(k't)) \geq \mu(\exp(k't)) = \exp(\exp(\frac{1}{2}k't)) \geq \exp\left(\frac{k'}{\varepsilon}\exp(t^{1/2})\right) = \exp\left(\frac{k'}{\varepsilon}\phi(t)\right),$$

for t large enough, and so (3.5.11) holds for these values of t .

Now suppose that $t \in [t_{n+1}^{3/4}, t_{n+1}]$, for some $n \in \mathbb{N}$. Then, for $k' > 1$,

$$\begin{aligned} \phi(\exp(k't)) &\geq \phi(\exp(k't_{n+1}^{3/4})) \geq \mu(\exp(k't_{n+1}^{3/4})) \\ &= \exp(\exp(\frac{1}{2}k't_{n+1}^{3/4})) \\ &\geq \exp\left(\frac{k'}{\varepsilon}\exp(t_{n+1}^{1/2})\right), \quad \text{for } t_{n+1} \text{ large enough,} \\ &= \exp\left(\frac{k'}{\varepsilon}\phi(t_{n+1})\right) \\ &\geq \exp\left(\frac{k'}{\varepsilon}\phi(t)\right), \end{aligned}$$

and hence (3.5.11) is satisfied.

Now, Lemma 3.13 implies that ψ satisfies (3.5.10) which means that f is 2-log-regular. □

CHAPTER 4

Uniformly escaping spiders' webs

4.1. Introduction

In addition to looking at sets of uniform rates of escape and checking when they are equal to $A(f)$, we can also use some sets of uniform rates of escape as a tool in proving several interesting results concerning spiders' webs and non-escaping endpoints. This is the focus of the next two chapters. In this chapter we focus on spiders' webs and, in particular, we show that for Fatou's function $f(z) = z + 1 + e^{-z}$, the escaping set has the structure of a spider's web. Recall from §1.1.5 that for Fatou's function $I(f)$ consists of all the points in the complex plane except for some of the endpoints of the curves in $J(f)$ (which form a Cantor bouquet), and $I(f)$ is connected. The following theorem shows that it is in fact a spider's web.

THEOREM 4.1. *Let $f(z) = z + 1 + e^{-z}$. Then $I(f)$ is a spider's web.*

Theorem 4.1 gives a positive answer to a question of Rippon and Stallard ([21, Problem 9]) and has some interesting consequences discussed in Section 5.2. Note that $A(f)$ is not a spider's web since it consists of the curves in $J(f)$ except for some of their endpoints and so it is not even connected as mentioned in §1.3.1.

In fact we can prove a stronger result than Theorem 4.1 and in order to state this we consider points which escape to infinity at a uniform rate.

Let f be a transcendental entire function and let $(a_n)_{n \in \mathbb{N}}$ be a positive sequence such that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $I(f, (a_n))$ be the subset of $I(f)$ defined in Section 1.3, that is

$$I(f, (a_n)) = \{z \in \mathbb{C} : |f^n(z)| \geq a_n, \text{ for } n \in \mathbb{N}\}.$$

Now consider the sequence

$$\left(\frac{n+m}{2}\right) = \frac{1+m}{2}, \frac{2+m}{2}, \dots, \text{ where } m \in \mathbb{N}.$$

We have the following theorem.

THEOREM 4.2. *Let $f(z) = z + 1 + e^{-z}$. Then $I(f, ((n+m)/2))$ contains a spider's web, for all $m \in \mathbb{N}$.*

REMARK 4.1.1. A set related to $I(f, ((n+m)/2))$ was used by Bergweiler and Peter in [19] in relation to the Hausdorff measure for entire functions. More precisely, they considered the set

$$\text{Unb}(f, (p_n)) = \{z \in \mathbb{C} : |f^n(z)| > p_n, \text{ for infinitely many } n\},$$

where (p_n) is a real sequence tending to infinity.

Theorem 4.2 shows that $I(f)$ contains a spider's web and hence it is a spider's web (see [66, Lemma 4.5]), and so Theorem 4.1 follows.

An approximation to the set $I(f, ((n+m)/2))$, for $f(z) = z + 1 + e^{-z}$ and $m = 6$, is shown in Figure 4.1. The set of white points together with the light grey points in $J(f)$ is the complement of $I(f, ((n+6)/2))$. Note that, since the boundary of the largest visible complementary component of $I(f, ((n+6)/2))$ lies in $I(f, ((n+6)/2))$, Figure 4.1 shows a loop in $I(f)$ that surrounds some of the non-escaping endpoints of $J(f)$.

We prove the following general sufficient condition for $I(f)$ to be a spider's web, which can be used to prove Theorem 4.1.

THEOREM 4.3. *Let f be a transcendental entire function. If*

- $I(f, (a_n))$ is defined as above;
- the disc $D(0, a_n)$ contains a periodic cycle of f , for all $n \in \mathbb{N}$, and
- $I(f, (a_n))^c$ has a bounded component,

then $I(f)$ is a spider's web.

In order to prove Theorem 4.2 we need the following general result.

THEOREM 4.4. *Let f be a transcendental entire function. If*

- $I(f, (a_n))$ is defined as above;
- (a_n) is increasing;
- $a_{n+1} \leq M(a_n)$, for $n \in \mathbb{N}$, and $a_1 < M(a_1)$ and

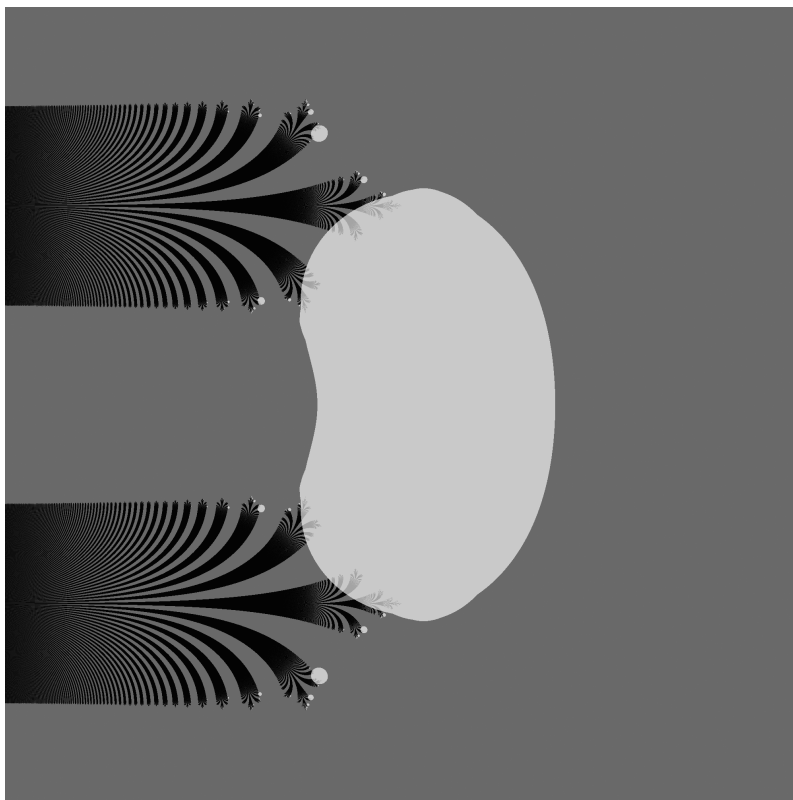


FIGURE 4.1. Complementary components of $I(f, ((n+6)/2))$ for $f(z) = z + 1 + e^{-z}$

- $I(f, (a_n))^c$ has a bounded component,

then $I(f)$ is a spider's web.

In Section 4.2 we give the proofs of Theorems 4.3 and 4.4. In Section 4.3 we prove some lemmas that we need in the proof of Theorem 4.2, which is given in Section 4.4.

In Section 4.5 we adapt the method from the proof of Theorem 4.2 in order to show that $I(f)$ is a spider's web for another well known function, namely, $f(z) = 2z + 2 - \log 2 - e^z$. (This function was first studied by Bergweiler in [10] who showed that it has a Baker domain that contains a left half-plane and also has several other interesting dynamical properties. Note that our result implies that $I(f)$ is a connected set in this case, a property that was not known before.)

The same methods of proof can be used to show that the escaping set is a spider's web for two families of functions each of which behaves in a similar way to one of the two functions we study here:

REMARK 4.1.2. Let $\{f : f(z) = az + b + ce^{dz}\}$, where $a, b, c, d \in \mathbb{R}$, be a family of transcendental entire functions such that:

- (a) $a = 1$, $c \neq 0$, $bd < 0$; or
- (b) $a > 1$, $c, d \neq 0$.

Then similar arguments to those used in Section 4.4 and Section 4.5 show that for the families of functions defined in (a) and (b), respectively, $I(f)$ is a spider's web. Note that within each of the families in (a) and (b), there are many examples of functions which are conjugate to each other under linear transformations of \mathbb{C} .

Finally in Section 4.6 we give another general result on spiders' webs which can be used to show that, for some functions, $I(f) \cup F(f)$ contains a spider's web (Corollary 4.15).

4.2. Results on spiders' webs

In this section we give the proofs of Theorems 4.3 and 4.4.

PROOF OF THEOREM 4.3. Let G be a bounded component of $I(f, (a_n))^c$. As $I(f, (a_n))$ is a closed set, we deduce that G is open and

$$(4.2.1) \quad \partial G \subset I(f, (a_n)) \subset I(f).$$

Now let $G_n = \widetilde{f^n(G)}$, where \widetilde{U} denotes the union of U and its bounded complementary components. (The notation $T(U)$ is often used to describe the set \widetilde{U} —see for example [9], [14]). We show that there exists a sequence (n_k) such that

$$(4.2.2) \quad G_{n_k} \subset G_{n_{k+1}} \text{ and } \partial G_{n_k} \subset I(f), \text{ for } k \in \mathbb{N}, \text{ and } \bigcup_k G_{n_k} = \mathbb{C}.$$

First note that, for each $n \in \mathbb{N}$, $\partial G_n \subset f^n(\partial G)$, since G is a bounded domain. Also, as $\partial G \subset I(f, (a_n))$, we have that

$$f^n(\partial G) \subset \{z : |z| \geq a_n\}$$

and hence

$$(4.2.3) \quad \partial G_n \subset \{z : |z| \geq a_n\}, \text{ for } n \in \mathbb{N}.$$

In order to show that ∂G_n surrounds the disc $D(0, a_n)$ for n large enough, note that $G \subset I(f, (a_n))^c$ and so, for each point z in G , there exists $N \in \mathbb{N}$ such that $|f^N(z)| < a_N$. It follows from (4.2.3) that ∂G_N surrounds $D(0, a_N)$. Since the disc $D(0, a_n)$ contains a periodic cycle, for all $n \in \mathbb{N}$, it then follows from (4.2.3) that

$$(4.2.4) \quad G_n \supset D(0, a_n), \text{ for all } n \geq N.$$

Note now that, for $n \in \mathbb{N}$,

$$(4.2.5) \quad \partial G_n \subset f^n(\partial G) \subset I(f),$$

by (4.2.1). It follows from (4.2.4) and (4.2.5), together with the fact that $a_n \rightarrow \infty$ as $n \rightarrow \infty$, that we can find a sequence (n_k) such that (G_{n_k}) satisfies (4.2.2).

We know that $I(f)$ has at least one unbounded component, I_0 say (see [70, Theorem 1]). Clearly I_0 must meet ∂G_{n_k} for all sufficiently large k . Hence, by (4.2.2), there exists a subset of $I(f)$ which is a spider's web and this implies that $I(f)$ is a spider's web (see [66, Lemma 4.5]). \square

PROOF OF THEOREM 4.4. Let G be a bounded component of $I(f, (a_n))^c$ and $G_n = \widetilde{f^n(G)}$ as before. In addition to the fact proved in the proof of Theorem 4.3, we also need to show that $\partial G_m \subset I(f, (a_n))$, for $m \in \mathbb{N}$, and that $I(f, (a_n))$ has an unbounded component. As (a_n) is increasing and $\partial G \subset I(f, (a_n))$, we have

$$\partial G_m \subset f^m(\partial G) \subset I(f, (a_{n+m})) \subset I(f, (a_n)), \text{ for } m \in \mathbb{N}.$$

Hence, the subsequence (G_{n_k}) found in the proof of Theorem 4.3 satisfies $\partial G_{n_k} \subset I(f, (a_n))$ in addition to (4.2.2). Note that, since (a_n) is increasing, passing to a subsequence if necessary, we can again assume that $D(0, a_n)$ contains a periodic cycle for all $n \in \mathbb{N}$.

Recall now that the set $A_R(f)$, which is discussed in §1.3.1, has the property that each of its components is unbounded, whenever $R > 0$ is such that $M(r, f) > r$ for $r \geq R$ (see [68, Theorem 1.1]). Without loss of generality take $R > a_1$ and so $M(R, f) \geq a_1$, and let $z \in A_R(f)$ ($A_R(f) \neq \emptyset$ by [68, Theorem 2.5]). Then $|f^n(z)| \geq M^n(R)$, for $n \in \mathbb{N}$. As

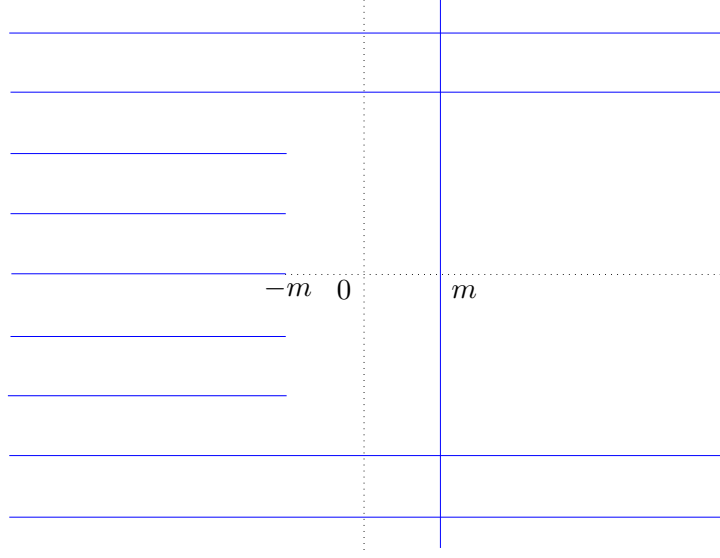


FIGURE 4.2. The sets described in Lemma 4.5

$a_{n+1} \leq M(a_n)$, for all $n \in \mathbb{N}$, and $a_1 < M(a_1)$, we deduce that $M^n(R) \geq a_n$, for all $n \in \mathbb{N}$, and so $z \in I(f, (a_n))$. Therefore, $I(f, (a_n))$ has at least one unbounded component. This will meet all ∂G_{n_k} , for k large enough, and hence $I(f, (a_n))$ contains a spider's web. \square

4.3. Preliminary lemmas

For $m \in \mathbb{N}$ and $(a_n)_{n \in \mathbb{N}} = (\frac{n+m}{2})_{n \in \mathbb{N}}$, the definition of $I(f, (a_n))$ is

$$I(f, ((n+m)/2)) = \{z \in \mathbb{C} : |f^n(z)| \geq (n+m)/2, n \in \mathbb{N}\}, \text{ for } m \in \mathbb{N}.$$

In this section we prove some basic results concerning the function f , defined by $f(z) = z + 1 + e^{-z}$, and the sets $I(f, ((n+m)/2))$, which we will use in the proof of Theorem 4.2. We first show that each set $I(f, ((n+m)/2))$, $m \in \mathbb{N}$, contains a right half-plane as well as a family of horizontal half-lines and a family of horizontal lines.

LEMMA 4.5. *Let $f(z) = z + 1 + e^{-z}$. For $m \in \mathbb{N}$, $I(f, ((n+m)/2))$ contains the following sets:*

- (1) *the right half-plane $\{x + yi : x \geq m\}$;*
- (2) *the half-lines of the form $\{x + 2j\pi i : x \leq -m\}$, for each $j \in \mathbb{Z}$ with $|j| < m$; and*

(3) the lines of the form $\{x + 2j\pi i : x \in \mathbb{R}\}$, for each $j \in \mathbb{Z}$ with $|j| \geq m$.

PROOF. Case (1): If $z = x + yi$ and $\operatorname{Re}(z) = x \geq m$, then

$$\operatorname{Re}(f(z)) = x + 1 + e^{-x} \cos y \geq x + \frac{1}{2},$$

since $e^{-x} < 1/2$, for $x \geq 1$. Thus, $\operatorname{Re}(f^n(z)) \geq m + n/2 > (n + m)/2$, for $n \in \mathbb{N}$, and so $z \in I(f, ((n + m)/2))$.

Case (2): Let $z = x + yi$ with $x \leq -m$ and $y = 2k\pi i$, where $k \in \mathbb{Z}$ and $|k| < m$. Then

$$\operatorname{Re}(f(z)) = x + 1 + e^{-x} \geq -m + 1 + e^m > m + 1,$$

since $g(x) = x + 1 + e^{-x}$ is decreasing for $x < 0$. Hence, by the argument in case (1), $z \in I(f, ((n + m)/2))$.

Case (3): Let $z_0 = 2m\pi i$. Then, since the lines in (3) are invariant under f , for $1 \leq n \leq m$, we have

$$|f^n(z_0)| > \operatorname{Im}(f^n(z_0)) = 2m\pi \geq (m + n)\pi > \frac{n + m}{2}.$$

Also, for $n > m$, we have

$$|f^n(z_0)| > \operatorname{Re}(f^n(z_0)) > n > \frac{n + m}{2}.$$

Hence $z_0 \in I(f, ((n + m)/2))$. For any other point of the form $z_1 = x + 2m\pi i$, $x \neq 0$, we have $f(z_1) = x' + 2m\pi i$, where $x' > 2$. Hence $\operatorname{Im}(f^n(z_1)) = \operatorname{Im}(f^n(z_0)) = 2m\pi$. Since $f(x) = x + 1 + e^{-x}$ is increasing for $x > 0$ and $x' > \operatorname{Re}(f(z_0))$, we deduce that $\operatorname{Re}(f^n(z_1)) > \operatorname{Re}(f^n(z_0))$. Therefore, $z \in I(f, ((n + m)/2))$, for $z = x + 2m\pi i$, $x \in \mathbb{R}$.

Now take $z_2 = x + 2k\pi i$, where $k \in \mathbb{Z}$ with $|k| \geq m$ and $x \in \mathbb{R}$. Then

$$\operatorname{Im}(f^n(z_2)) \geq 2m\pi = \operatorname{Im}(f^n(x + 2m\pi i)), \text{ and } \operatorname{Re}(f^n(z_2)) = \operatorname{Re}(f^n(x)),$$

so we deduce that $z \in I(f, ((n + m)/2))$, for all $z = x + 2k\pi i$, $k \in \mathbb{Z}$ with $|k| \geq m$. □

Now fix $m \in \mathbb{N}$. Next we define the rectangles

$$R_k = \{x + yi : |x| < m + k, |y| < 2(m + k)\pi\}, \quad k \geq 0.$$

Note that

$$(4.3.1) \quad R_k \subset \{z : |z| < 3\pi(m+k)\},$$

since, for $z \in R_k$,

$$|z| \leq \sqrt{(m+k)^2 + 4\pi^2(m+k)^2} < 3\pi(m+k).$$

We also define the horizontal half-strips

$$S_{k,j} = \begin{cases} \{x + yi : x \leq -m - k, 2j\pi \leq y \leq 2(j+1)\pi\}, & \text{if } -m - k \leq j < m + k, \\ \{x + yi : x \leq m + k, 2j\pi \leq y \leq 2(j+1)\pi\}, & \text{otherwise,} \end{cases}$$

where $k \geq 0$ and $j \in \mathbb{Z}$. We have the following lemma concerning the rectangles R_k .

LEMMA 4.6. *Let $f(z) = z + 1 + e^{-z}$. If $z \in I(f, ((n+m+k)/2))^c$, where $m \in \mathbb{N}, k \geq 0$ are fixed, and $f(z)$ lies outside R_{k+1} , then $f(z)$ lies in some half-strip $S_{k+1,j}$, $j \in \mathbb{Z}$.*

PROOF. Note first that for $m \in \mathbb{N}$ and $k \geq 0$, if $f(z) \in I(f, ((n+m+k+1)/2))$ and $|f(z)| \geq (m+k+1)/2$, then $z \in I(f, ((n+m+k)/2))$. Hence,

$$\text{if } z \in I(f, ((n+m+k)/2))^c \quad \text{and} \quad |f(z)| \geq (m+k+1)/2,$$

$$\text{then } f(z) \in I(f, ((n+m+k+1)/2))^c.$$

Now assume that $z \in I(f, ((n+m+k)/2))^c$ and $f(z)$ lies outside R_{k+1} . Then $|f(z)| \geq (m+k+1)/2$ and hence $f(z) \in I(f, ((n+m+k+1)/2))^c$. Therefore, we deduce by Lemma 4.5, with m replaced by $m+k+1$, that $f(z)$ lies in some half-strip of the form $S_{k+1,j}$, $j \in \mathbb{Z}$. \square

The last result of this section gives some basic estimates for the size of $|f|$ in certain sets.

LEMMA 4.7. *Let $f(z) = z + 1 + e^{-z}$. If $-\operatorname{Re}(z) \geq |z|/2$ and $-\operatorname{Re}(z) \geq 3$, then*

$$(4.3.2) \quad \frac{1}{2}e^{-\operatorname{Re}(z)} \leq |f(z)| \leq 2e^{-\operatorname{Re}(z)}.$$

PROOF. In order to prove the first inequality of (4.3.2) note that

$$\begin{aligned}
 |f(z)| &\geq |e^{-z}| - 1 - |z| \\
 &= e^{-\operatorname{Re}(z)} - 1 - |z| \\
 &\geq e^{-\operatorname{Re}(z)} - 1 + 2\operatorname{Re}(z) \\
 &\geq \frac{1}{2}e^{-\operatorname{Re}(z)},
 \end{aligned}$$

since $e^t/2 \geq 1 + 2t$, for $t \geq 3$.

For the second inequality of (4.3.2) we have

$$\begin{aligned}
 |f(z)| &\leq |e^{-z}| + 1 + |z| \\
 &= e^{-\operatorname{Re}(z)} + 1 + |z| \\
 &\leq e^{-\operatorname{Re}(z)} + 1 - 2\operatorname{Re}(z) \\
 &\leq 2e^{-\operatorname{Re}(z)},
 \end{aligned}$$

since $e^t \geq 1 + 2t$, for $t \geq 2$. □

4.4. Fatou's web

In this section we prove Theorem 4.2 which states that, for Fatou's function f , the set $I(f, (n+m)/2)$ contains a spider's web for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$ be fixed. The idea of the proof of Theorem 4.2 is to show first that all the components of $I(f, ((n+m)/2))^c$ are bounded and then use Theorem 4.4. We are going to assume that there exists an unbounded component of $I(f, ((n+m)/2))^c$ and obtain a contradiction. In order to obtain the contradiction we will make use of the following lemma (see [72, Lemma 1]).

LEMMA 4.8. *Let $E_n, n \geq 0$, be a sequence of compact sets in \mathbb{C} and $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a continuous function such that*

$$f(E_n) \supset E_{n+1}, \quad \text{for } n \geq 0.$$

Then there exists $\zeta \in E_0$ such that

$$f^n(\zeta) \in E_n, \quad \text{for } n \geq 0.$$

PROOF OF THEOREM 4.2. Suppose that there exists an unbounded component U of the complement of $I(f, ((n+m)/2))$. We will construct a certain bounded curve $\gamma_0 \subset U$ and by considering the images $f^k(\gamma_0)$, $k \geq 0$, we will deduce, by Lemma 4.8, that γ_0 contains a point of $I(f, ((n+m)/2))$, giving a contradiction. As $I(f, ((n+m)/2))$ is closed, U is open and hence, by Lemma 4.5, there exists $j_0 \in \mathbb{Z}$ such that U contains $z_{0,1}, z_{0,2} \in S_{0,j_0}$, where $S_{0,j}$, $j \in \mathbb{Z}$, is a half-strip defined in Section 4.3, and a curve $\gamma_0 \subset U$ joining $z_{0,1}$ to $z_{0,2}$ such that:

- (a) $\operatorname{Re}(z_{0,1}) = a_0 < -3(m+1)$;
- (b) $\operatorname{Re}(z_{0,2}) = a_0 - 10$;
- (c) $\gamma_0 \subset \{z : a_0 - 10 \leq \operatorname{Re}(z) \leq a_0\} \cap S_{0,j_0}$;
- (d) $-\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_0$.

We will use induction to show that there exists a family of curves γ_k , $k \geq 0$, with $\gamma_{k+1} \subset f(\gamma_k)$, such that, for all $k \geq 0$, γ_k lies in some S_{k,j_k} and joins the points $z_{k,1}, z_{k,2}$ such that:

- (a) $\operatorname{Re}(z_{k,1}) = a_k < -3(m+k+1)$;
- (b) $\operatorname{Re}(z_{k,2}) = a_k - 10$;
- (c) $\gamma_k \subset \{z : a_k - 10 \leq \operatorname{Re}(z) \leq a_k\} \cap S_{k,j_k}$;
- (d) $-\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_k$.

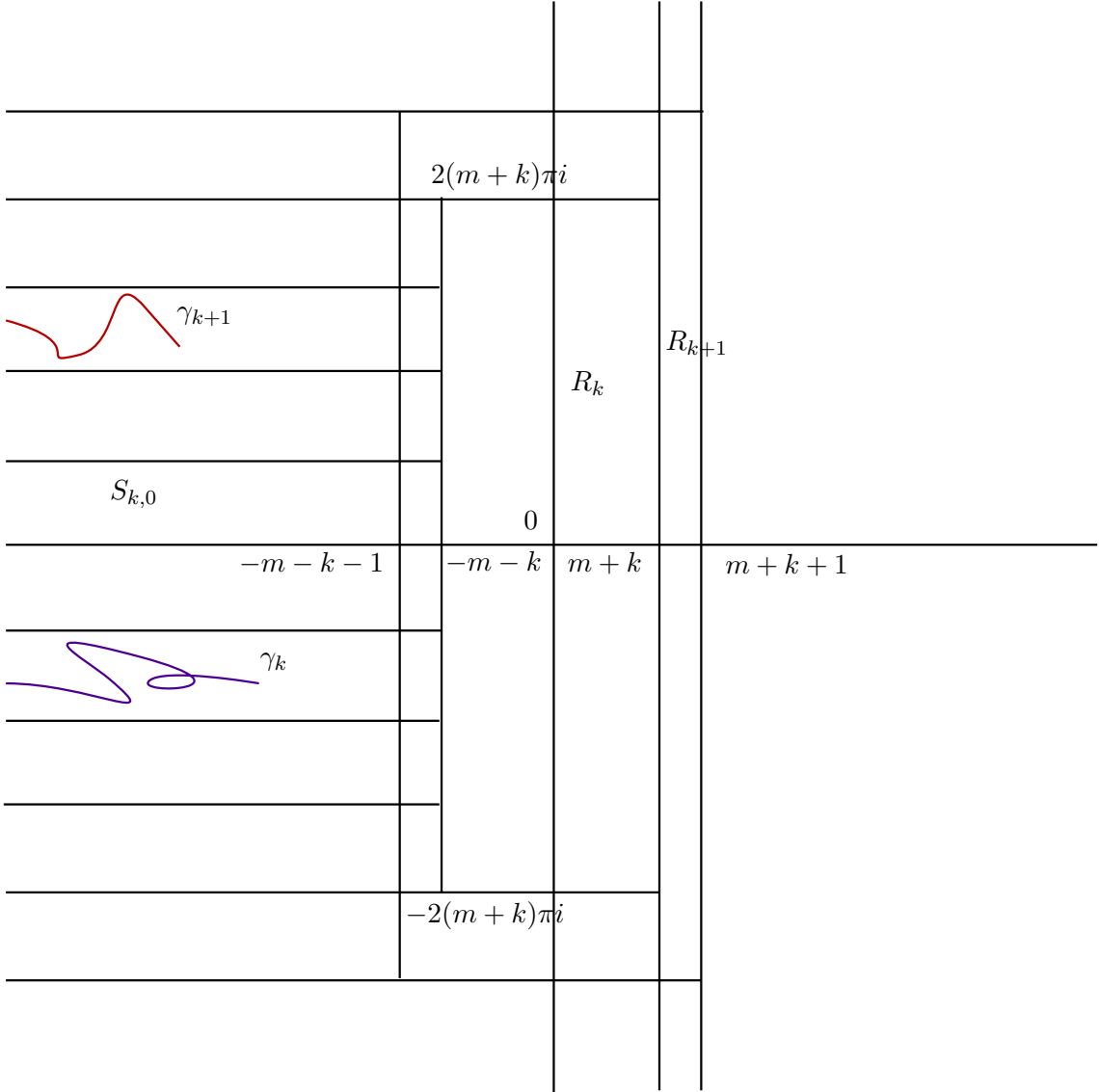
By assumption this is true for $k = 0$. Suppose that it holds for $0 \leq j \leq k$; we show that it is also true for $j = k+1$.

By (a), (c) and (d), and the left-hand inequality of (4.3.2), we deduce that, for $z \in \gamma_k$,

$$(4.4.1) \quad |f(z)| \geq e^{-a_k}/2 > 3(m+k+1)\pi.$$

Hence, by (4.3.1), the curve $f(\gamma_k)$ lies outside R_{k+1} and so, by Lemma 4.6, it must lie in some half-strip $S_{k+1,j_{k+1}}$. Now, by both inequalities of (4.3.2), we have

$$(4.4.2) \quad |f(z_{k,2})| \geq \frac{e^{-\operatorname{Re}(z_{k,2})}}{2} = \frac{e^{10}}{2} e^{-a_k} \geq \frac{e^{10}}{4} |f(z_{k,1})|.$$

FIGURE 4.3. The construction of the curves γ_k

Since $f(\gamma_k) \subset S_{k+1,j_{k+1}} \subset \{z : \operatorname{Re}(z) \leq m+k+1\}$, it follows from (4.4.1) and (4.4.2) by a routine calculation that the set

$$\left\{ w \in f(\gamma_k) : |w| \geq \frac{e^{10}}{8} |f(z_{k,1})| \right\}$$

contains points $z_{k+1,1}, z_{k+1,2}$ and a curve $\gamma_{k+1} \subset f(\gamma_k)$ (see Figure 4.2) joining $z_{k+1,1}$ to $z_{k+1,2}$ such that

- (a) $\operatorname{Re}(z_{k+1,1}) = a_{k+1} < -3(m+k+2)$;
- (b) $\operatorname{Re}(z_{k+1,2}) = a_{k+1} - 10$;
- (c) $\gamma_{k+1} \subset \{z : a_{k+1} - 10 \leq \operatorname{Re}(z) \leq a_{k+1}\} \cap S_{k+1,j_{k+1}}$;
- (d) $-\operatorname{Re}(z) > |z|/2$, for $z \in f(\gamma_k)$.

As $\gamma_{k+1} \subset f(\gamma_k)$, for $k \geq 0$, it follows from Lemma 4.8 that there exists $\zeta \in \gamma_0$ such that $f^k(\zeta) \in \gamma_k$, for all $k \geq 0$. Hence, by (4.4.1),

$$|f^k(\zeta)| > 3(k+m)\pi > \frac{k+m}{2}, \text{ for } k \geq 0,$$

and so $\zeta \in I(f, ((n+m)/2))$, which gives us a contradiction. Since $(2k+1)\pi i$, $k \in \mathbb{Z}$, are fixed points of f , $I(f, ((n+m)/2))^c \neq \emptyset$, and so we deduce that all the components of $I(f, ((n+m)/2))^c$ are bounded.

Now take $m \geq 6$. Since the disc $D(0, 7/2)$ contains two fixed points of f , at $\pm\pi i$, each disc $D(0, (n+m)/2)$, $n \in \mathbb{N}$, $m \geq 6$ contains two fixed points. Finally, as $a_{n+1} \leq M(a_n)$ for $a_n = (n+m)/2$, all the hypotheses of Theorem 4.4 are satisfied and so we deduce that $I(f, ((n+m)/2))$ contains a spider's web, for all $m \geq 6$. Since $I(f, ((n+m')/2)) \supset I(f, ((n+m)/2))$ whenever $m' < m$, it follows that $I(f, ((n+m)/2))$ contains a spider's web for all $m \in \mathbb{N}$. \square

REMARK 4.4.1. In the proof of Theorem 4.2 we can actually obtain a contradiction by supposing that $I(f, ((n+m)/2))^c$ has a component of sufficiently large diameter. In fact it follows from the proof that every component U of $I(f, ((n+m)/2))^c$ that lies outside the rectangle R_0 , and for which all $z \in U$ satisfy $-\operatorname{Re}(z) \geq |z|/2$, has diameter less than 12.

4.5. Bergweiler's web

By adapting the method used in Section 4.4, we can show that $I(f)$ is a spider's web for other transcendental entire functions, in particular some that have similar properties to the Fatou function. A well known function in this category is the function $f(z) = 2z + 2 - \log 2 - e^z$ that was first studied by Bergweiler in [10].

In order to produce a similar argument to that used to prove Theorem 4.2, we introduce the set

$$I(f, (2^{n/2})) = \{z \in \mathbb{C} : |f^n(z)| \geq 2^{n/2}, \text{ for } n \in \mathbb{N}\}.$$

We prove the following result.

THEOREM 4.9. *Let $f(z) = 2z + 2 - \log 2 - e^z$. Then $I(f, (2^{n/2}))$ contains a spider's web, and hence $I(f)$ is a spider's web.*

We first prove some lemmas similar to the ones given in Section 4.3 which we need in the proof of Theorem 4.9.

We consider the family of sets

$$I(f, (2^{(n+m)/2})) = \{z \in \mathbb{C} : |f^n(z)| \geq 2^{(n+m)/2}, \text{ for } n \in \mathbb{N}\}, \text{ for } m \geq 0.$$

LEMMA 4.10. *Let $f(z) = 2z + 2 - \log 2 - e^z$. For $m \geq 0$, $I(f, (2^{(n+m)/2}))$ contains the following sets:*

- (1) *the left half-plane $\{x + yi : x \leq -2^{m+2}\}$;*
- (2) *the half-lines of the form $\{x + 2j\pi i : x \geq 2^{m+2}\}$, for each $j \in \mathbb{Z}$ with $|j| < 2^m$;*
- and*
- (3) *the lines of the form $\{x + 2j\pi i : x \in \mathbb{R}\}$, for each $j \in \mathbb{Z}$ with $|j| \geq 2^m$.*

PROOF. Case (1): If $z = x + yi$ and $\operatorname{Re}(z) = x \leq -2^{m+2}$, then

$$\operatorname{Re}(f(z)) = 2x + 2 - \log 2 - e^x \cos y \leq 2x + 2 < -2^{m+2},$$

since $e^x < \log 2$, for $x \leq -1$. Hence

$$\operatorname{Re}(f^2(z)) \leq 2\operatorname{Re}(f(z)) + 2 \leq 2(2x + 2) + 2 \leq 2^2x + 6,$$

and, by induction, we have

$$\operatorname{Re}(f^n(z)) \leq 2^n x + 2^{n+1} - 2, \text{ for } n \in \mathbb{N}.$$

Thus, for $n \in \mathbb{N}$,

$$\operatorname{Re}(f^n(z)) \leq 2^n x + 2^{n+1} - 2 \leq -2^{n+m+2} + 2^{n+1} - 2 \leq -2^{(n+m)/2},$$

and so $z \in I(f, (2^{(n+m)/2}))$.

Case (2): Let $z = x + yi$ with $x \geq 2^{m+2}$ and $y = 2k\pi$, for some $k \in \mathbb{Z}$ with $|k| < 2^m$. For $m \geq 0$, we have $2^{m+2} \geq 4$ and

$$\operatorname{Re}(f(z)) = 2x + 2 - \log 2 - e^x \leq -\frac{e^x}{2} < -2x \leq -2^{m+3},$$

and so, by the argument in Case (1), $z \in I(f, (2^{(n+m)/2}))$.

Case (3): Let $z = x + 2k\pi i$, for some $k \in \mathbb{Z}$ with $|k| \geq 2^m$. Since $\operatorname{Im}(f(z)) = 2^2 k\pi \geq 2^2 2^m$, for $n \in \mathbb{N}$, we have $\operatorname{Im}(f^n(z)) \geq 2^{n+1} 2^m > 2^{(n+m)/2}$, and so $z \in I(f, (2^{(n+m)/2}))$. \square

We fix $m = 0$ and we define the rectangles

$$R_k = \{z = x + yi : |x| < 2^{k+2}, |y| < (2^{k+1})\pi\}, \quad k \geq 0.$$

Note that

$$(4.5.1) \quad R_k \subset \{z : |z| < 2^{k+3}\},$$

since, for $z \in R_k$,

$$|z| \leq \sqrt{4^{k+2} + \pi^2 4^{k+1}} < 2^{k+3}.$$

We also define the horizontal half-strips

$$S_{k,j} = \begin{cases} \{x + yi : x \geq 2^{k+2}, 2j\pi \leq y \leq 2(j+1)\pi\}, & \text{if } -2^k \leq j < 2^k, \\ \{x + yi : x \geq -2^{k+2}, 2j\pi \leq y \leq 2(j+i)\pi\}, & \text{otherwise,} \end{cases}$$

where $k \geq 0$ and $j \in \mathbb{Z}$.

We have the following lemma concerning the rectangles R_k .

LEMMA 4.11. *Let $f(z) = 2z + 2 - \log 2 - e^z$. If $z \in I(f, (2^{(n+k+1)/2}))^c$, where $k \geq 0$ is fixed, and $f(z)$ lies outside R_{k+1} , then $f(z)$ lies in some half-strip $S_{k+1,j}$, $j \in \mathbb{Z}$.*

PROOF. Note first that for any fixed $k \geq 0$, if $f(z) \in I(f, (2^{(n+k+1)/2}))$ and $|f(z)| \geq 2^{(k+1)/2}$, then $z \in I(f, (2^{(n+k)/2}))$. Hence,

$$\text{if } z \in I(f, (2^{(n+k)/2}))^c \text{ and } |f(z)| \geq 2^{(k+1)/2},$$

$$\text{then } f(z) \in I(f, (2^{(n+k+1)/2}))^c.$$

Now assume that $z \in I(f, (2^{(n+k)/2}))^c$ and $f(z)$ lies outside R_{k+1} . Then $|f(z)| \geq 2^{(k+1)/2}$ and hence $f(z) \in I(f, (2^{(n+k+1)/2}))^c$. Therefore, we deduce by Lemma 4.10 that $f(z)$ lies in some half-strip of the form $S_{k+1,j}$, $j \in \mathbb{Z}$. \square

Finally, we can prove the following basic estimate.

LEMMA 4.12. *Let $f(z) = 2z + 2 - \log 2 - e^z$. If $\operatorname{Re}(z) \geq |z|/2$ and $\operatorname{Re}(z) \geq 4$, then*

$$(4.5.2) \quad \frac{1}{2}e^{\operatorname{Re}(z)} \leq |f(z)| \leq 2e^{\operatorname{Re}(z)}.$$

PROOF. In order to prove the first inequality of (4.5.2) note that

$$\begin{aligned} |f(z)| &\geq |e^z| - |2z| - 2 + \log 2 \\ &= e^{\operatorname{Re}(z)} - 2|z| - 2 + \log 2 \\ &\geq e^{\operatorname{Re}(z)} - 4\operatorname{Re}(z) - 2 + \log 2 \\ &\geq \frac{1}{2}e^{\operatorname{Re}(z)}, \end{aligned}$$

since $e^t/2 \geq 4t + 2 - \log 2$, for $t \geq 4$.

For the second inequality of (4.5.2) we have

$$\begin{aligned} |f(z)| &\leq |e^z| + 2|z| + 2 + \log 2 \\ &= e^{\operatorname{Re}(z)} + 2|z| + 2 + \log 2 \\ &\leq e^{\operatorname{Re}(z)} + 4\operatorname{Re}(z) + 2 + \log 2 \\ &\leq 2e^{\operatorname{Re}(z)}, \end{aligned}$$

since $e^t \geq 4t + 2 + \log 2$, for $t \geq 3$. \square

Now we are ready to present an argument similar to the one we used in the proof of Theorem 4.2 and prove Theorem 4.9.

PROOF OF THEOREM 4.9. We show that all the hypotheses of Theorem 4.4 are satisfied and then the result will follow. The disc $D(0, 2^{n/2})$ contains the fixed point at $\log 2$ for all $n \in \mathbb{N}$. Hence, it suffices to show that $I(f, (2^{n/2}))^c$ has no unbounded components.

Suppose that there exists an unbounded component U of the complement of $I(f, (2^{n/2}))$. As before, we construct a curve $\gamma_0 \subset U$ and by considering the images $f^k(\gamma_0)$, $k \geq 0$, we deduce, by Lemma 4.8, that γ_0 contains a point of $I(f, (2^{n/2}))$, giving a contradiction. More precisely, as $I(f, (2^{n/2}))$ is closed, U is open and hence, by Lemma 4.10, there exists $j_0 \in \mathbb{Z}$ such that U contains $z_{0,1}, z_{0,2} \in S_{0,j_0}$ and a curve $\gamma_0 \subset U$ joining $z_{0,1}$ to $z_{0,2}$ such that:

- a) $\operatorname{Re}(z_{0,1}) = a_0 > 4$;
- b) $\operatorname{Re}(z_{0,2}) = a_0 + 10$;
- c) $\gamma_0 \subset \{z : a_0 \leq \operatorname{Re}(z) \leq a_0 + 10\} \cap S_{0,j_0}$;
- d) $\operatorname{Re}(z) \geq |z|/2$ for any $z \in \gamma_0$.

We will use induction to show that there exists a family of curves γ_k , $k \geq 0$, with $\gamma_{k+1} \subset f_a(\gamma_k)$, such that, for all $k \geq 0$, γ_k lies in some S_{k,j_k} and joins the points $z_{k,1}, z_{k,2}$ such that:

- a) $\operatorname{Re}(z_{k,1}) = a_k > 2^{k+2}$;
- b) $\operatorname{Re}(z_{k,2}) = a_k + 10$;
- c) $\gamma_k \subset \{z : a_k \leq \operatorname{Re}(z) \leq a_k + 10\} \cap S_{k,j_k}$;
- d) $\operatorname{Re}(z) \geq |z|/2$ for any $z \in \gamma_k$.

By assumption this is true for $k = 0$. Suppose that it holds for $0 \leq j \leq k$; we show that it is also true for $j = k + 1$.

By (a), (c) and (d), and the left-hand inequality of (4.5.2) we deduce that, for $z \in \gamma_k$,

$$(4.5.3) \quad |f(z)| \geq e^{a_k}/2 > 2^{k+4}.$$

Hence, by (4.5.1), $f(\gamma_k)$ lies outside R_{k+1} and so, by Lemma 4.11, it must lie in some half-strip $S_{k+1,j_{k+1}}$. Now, by both inequalities of (4.5.2) and (b), we have

$$(4.5.4) \quad |f(z_{k,2})| \geq \frac{e^{\operatorname{Re}(z_{k,2})}}{2} = \frac{e^{10}}{2} e^{a_k} \geq \frac{e^{10}}{4} |f(z_{k,1})|.$$

Since $f(\gamma_k) \subset S_{k+1,j_{k+1}} \subset \{z : \operatorname{Re}(z) \geq -2^{k+3}\}$, it follows from (4.5.3) and (4.5.4) by a routine calculation that the set

$$\left\{ w \in f(\gamma_k) : |w| \geq \frac{e^{10}}{8} |f(z_{k,1})| \right\}$$

contains points $z_{k+1,1}, z_{k+1,2}$ and a curve $\gamma_{k+1} \subset f(\gamma_k)$ joining them such that

- a) $\operatorname{Re}(z_{k+1,1}) = a_{k+1} > 2^{k+3}$;
- b) $\operatorname{Re}(z_{k+1,2}) = a_{k+1} + 10$;
- c) $\gamma_{k+1} \subset \{z : a_{k+1} \leq \operatorname{Re}(z) \leq a_{k+1} + 10\} \cap S_{k+1,j_{k+1}}$;
- d) $\operatorname{Re}(z) > |z|/2$, for any $z \in f(\gamma_k)$.

As $\gamma_{k+1} \subset f(\gamma_k)$, for $k \geq 0$, it follows from Lemma 4.8 that there exists $\zeta \in \gamma_0$ such that $f^k(\zeta) \in \gamma_{k+1}$, for all $k \geq 0$. Hence, by (4.5.3)

$$|f^k(\zeta)| > 2^{k+4} > 2^{k/2},$$

and so $\zeta \in I(f, (2^{n/2}))$, which gives us a contradiction. Therefore, all the components of $I(f, (2^{n/2}))^c$ are bounded.

Therefore, since $a_{n+1} \leq M(a_n)$, for $a_n = 2^{n/2}$, all the hypotheses of Theorem 4.4 are satisfied. Hence Theorem 4.4 implies that $I(f, (2^{n/2}))$ contains a spider's web and so, by [66, Lemma 4.5], $I(f)$ is a spider's web. \square

4.6. General theorem on spiders' webs

In the last section of this chapter, we prove a general result on spiders' webs from which we deduce that, for many functions, $I(f) \cup F(f)$ contains a spider's web. First we give a necessary definition following [38].

DEFINITION 4.13. Let U be a domain in \mathbb{C} and let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a smooth increasing function such that $\psi(0) = 0$ and $\psi(r) \leq r$, for all $r > 0$. We say that U is ψ -*LC-2* if, for every $z \in \mathbb{C}$, each pair of points in $U \setminus D(z, r)$ can be joined by an arc in $U \setminus D(z, \psi(r))$.

REMARK 4.6.1. When $\psi(r) = r/c$, $c > 1$, the above definition reduces to the more common *LLC-2* which is associated to *linear local connectedness*, a concept introduced by Gehring and Väisälä in [37] and first defined in [36].

We also need the following notation: we denote by U_{unb} the union of all the unbounded components of an open set U .

THEOREM 4.14. *Let f be a transcendental entire function and let U be an open set invariant under f , all of whose unbounded components are bounded by Jordan curves in $\hat{\mathbb{C}}$, and such that the following conditions are satisfied:*

- *there exists $\psi(r)$ such that all the components of U_{unb} are ψ -LC-2;*
- *there exists $r_0 > 0$ such that the disc $D(0, r_0)$ contains all the bounded components of \bar{U} ;*
- *there is a positive, increasing sequence (a_n) with $a_1 > r_0$ such that $a_n \rightarrow \infty$ as $n \rightarrow \infty$, $a_{n+1} \leq M(a_n)$ and $a_1 < M(a_1)$;*
- *$D(0, a_1) \setminus \bar{U}_{\text{unb}}$ contains a periodic cycle of f .*

Now consider

$$X = \{z \in \mathbb{C} : \forall n \in \mathbb{N}, |f^n(z)| \geq a_n \text{ or } f^n(z) \in \bar{U}\}.$$

If X^c has a bounded component, then X contains a spider's web.

REMARK 4.6.2. This result extends Theorem 4.4 since X contains $I(f, (a_n))$ and also contains points which do not need to escape. Theorem 4.4 is a special case of Theorem 4.14 with $U = \emptyset$.

PROOF. Let G be a bounded component of X^c . As X is a closed set, we deduce that G is open and

$$(4.6.1) \quad \partial G \subset X.$$

Now let $G_n = \widetilde{f^n(G)}$. We show that there exist domains G'_n with $\partial G'_n \subset \partial G_n \cup U$ and a sequence (n_k) such that

$$(4.6.2) \quad G'_{n_k} \subset G'_{n_{k+1}} \text{ and } \partial G'_{n_k} \subset X, \text{ for } k \in \mathbb{N}, \text{ and } \bigcup_k G'_{n_k} = \mathbb{C}.$$

First note that, for each $n \in \mathbb{N}$, $\partial G_n \subset f^n(\partial G)$, since G is a bounded domain. Also, as $\partial G \subset X$, we have that

$$f^n(\partial G) \subset \{z : |z| \geq a_n\} \cup \bar{U}.$$

Since $D(0, a_1)$ contains the bounded components of \bar{U} we deduce that

$$(4.6.3) \quad \partial G_n \subset f^n(\partial G) \subset A_n = \{z : |z| \geq a_n\} \cup \bar{U}_{\text{unb}}.$$

In order to show that ∂G_n surrounds $D(0, a_n) \setminus \overline{U}_{\text{unb}}$ for n large enough, note that $G \subset X^c$ and so, for each point z in G , there exists $N \in \mathbb{N}$ such that $|f^N(z)| < a_N$ and $f^N(z) \in \overline{U}^c$. It follows from (4.6.3) that ∂G_N surrounds $D(0, a_n) \setminus \overline{U}_{\text{unb}}$. Note that $D(0, a_n) \setminus \overline{U}_{\text{unb}}$ cannot be disconnected since all the components of U_{unb} are ψ -LC-2. Since $D(0, a_n) \setminus \overline{U}_{\text{unb}}$ contains a periodic cycle, for all $n \in \mathbb{N}$, it then follows from (4.6.3) that

$$(4.6.4) \quad G_n \supset D(0, a_n) \setminus \overline{U}_{\text{unb}}, \quad \text{for all } n \geq N$$

(see also Figure 4.3). Note now that, for $m \in \mathbb{N}$,

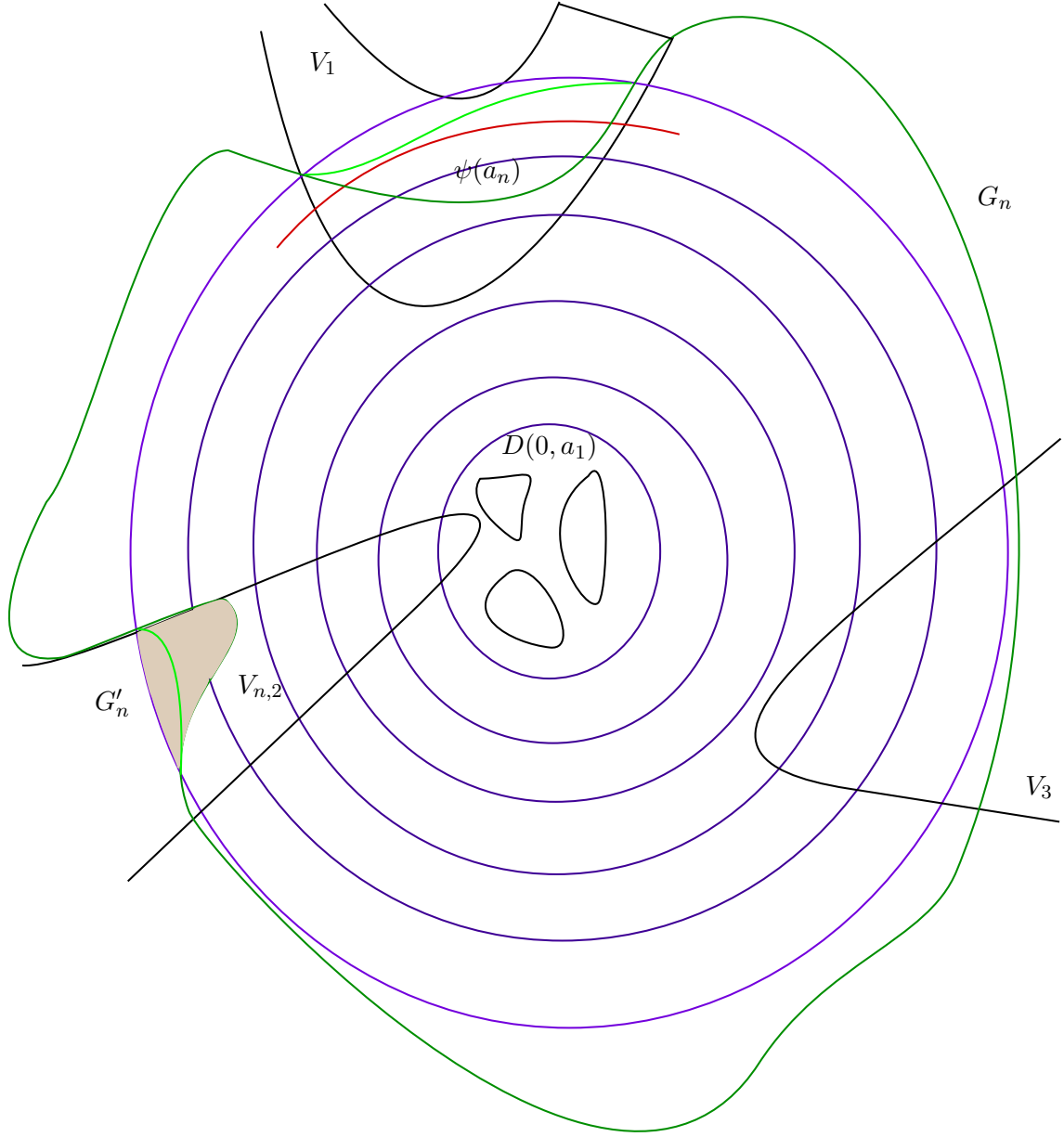
$$(4.6.5) \quad \begin{aligned} \partial G_m \subset f^m(\partial G) &= \{f^m(w) : w \in \partial G\} \\ &\subset \{z \in \mathbb{C} : \forall n \in \mathbb{N}, |f^n(z)| \geq a_{n+m} \text{ or } f^n(z) \in \overline{U}\} \\ &\subset X, \end{aligned}$$

as (a_n) is increasing.

Now consider $G_n^c \cap \overline{D(0, a_n)}$. For every connected component $V_{n,j}$, $j \in \mathbb{N}$, of $G_n^c \cap \overline{D(0, a_n)} \cap V_j$, where V_j is a component of U_{unb} , we take the pair of points lying in $\partial G_n^c \cap \partial D(0, a_n) = \partial G_n \cap \partial D(0, a_n)$ that lie in $\overline{V_{n,j}}$, say $z_{n,j}, z'_{n,j}$. Note that there is only one such pair of points for every connected component $V_{n,j}$ since G_n has no bounded complementary components. Also note that $V_{n,j} \subset V_j$. By hypothesis V_j is ψ -LC-2, and since its boundary is a Jordan curve in $\hat{\mathbb{C}}$ we can extend ψ -LC-2 to $\overline{V_j}$. Thus we can join the two points $z_{n,j}$ and $z'_{n,j}$ by an arc $\gamma_{n,j} \subset \overline{V_j} \setminus D(0, a_{\psi(a_n)})$. We take the bounded simply connected domain G'_n such that $\partial G'_n = (\partial G_n \cap \overline{D(0, a_n)}) \cup \bigcup_{n,j} \gamma_{n,j}$. Hence, $\partial G'_n \subset \partial G_n \cup \overline{U} \subset X$ and $\partial G'_n$ surrounds the disc $D(0, \psi(a_n))$, for all $n \geq N$. Since $\psi(r)$ is an increasing function, it follows that there exists a sequence (n_k) such that (4.6.2) is true.

Assume that $R > a_1$ and so $M(R, f) \geq a_1$, and let $z \in A_R(f)$ (the set $A_R(f)$ was defined in §1.3.1). Then $|f^n(z)| \geq M^n(R)$, for $n \in \mathbb{N}$. As $a_{n+1} \leq M(a_n)$, for all $n \in \mathbb{N}$, we deduce that $M^n(R) \geq a_n$, for all $n \in \mathbb{N}$, and so $z \in X$. Therefore, since all the components of $A_R(f)$ are unbounded ([68, Theorem 1.1]), X has at least one unbounded component. This will meet all $\partial G'_{n_k}$, for k large enough, and hence X contains a spider's web. \square

We now apply Theorem 4.14 to show that, for many functions, there is a spider's web in $I(f) \cup F(f)$.

FIGURE 4.4. The domains G_n and G'_n

COROLLARY 4.15. *Let f be a transcendental entire function and let all the hypotheses of Theorem 4.14 hold, with $\bar{U} \subset F(f)$. Then $F(f) \cup I(f, (a_n))$ contains a spider's web.*

PROOF. First recall that

$$X = \{z \in \mathbb{C} : \forall n \in \mathbb{N}, |f^n(z)| \geq a_n \text{ or } f^n(z) \in \bar{U}\}.$$

If $z \in X$ then either $z \in I(f, a_n)$ or there exists $N \in \mathbb{N}$ such that $f^N(z) \in \overline{U} \subset F(f)$ and hence $z \in F(f)$. Thus $X \subset F(f) \cup I(f, (a_n))$. Hence the result follows from Theorem 4.14. \square

REMARK 4.6.3. If (a_n) is chosen so that $I(f, (a_n)) \subset A(f)$, then Corollary 4.15 gives us a spider's web in $A(f) \cup F(f)$. We construct spiders' webs of this type in the next chapter using a different technique.

CHAPTER 5

Non-escaping endpoints of entire functions

5.1. Introduction

As we mentioned in Section 1.5, there is an interesting connection between spiders' webs and non-escaping endpoints. In this chapter we present results on non-escaping endpoints for Fatou's function and some functions in the exponential family using two different techniques.

In Section 5.2 we discuss the first result which is a consequence of Theorem 4.1. The fact that $I(f)$ is a spider's web for Fatou's function means that we can create loops in $I(f)$ which surround the non-escaping endpoints of $J(f)$. Hence we are able to show that the set of non-escaping endpoints together with infinity is a totally separated set.

Following the results of Mayer and Alhabib and Rempe-Gillen discussed in Section 1.5 it is natural to ask whether the same property holds for the set of non-escaping endpoints for functions in the exponential family. In Section 5.3, we show that for $f_a(z) = e^z + a$, $a \in F(f_a)$, the non-escaping endpoints cannot 'explode' since their union with infinity is itself a totally separated set. More precisely, we have the following result.

THEOREM 5.1. *Let $f_a(z) = e^z + a$, $a \in F(f_a)$. The set of non-escaping endpoints of f_a together with infinity is a totally separated set.*

In fact, we prove the stronger result that the union of the *non-fast* endpoints with infinity is a totally separated set for $f_a(z) = e^z + a$, $a \in F(f_a)$ (Theorem 5.6). The non-fast endpoints are the endpoints that do not belong to the fast escaping set $A(f)$, which was introduced in §1.3.1. Since the set of non-escaping endpoints is a subset of the set of non-fast endpoints it is also totally separated. Note that the result of Alhabib and Rempe-Gillen (Theorem 1.19) actually holds for the endpoints that belong to $A(f_a)$ (see [2, Remark after Theorem 1.4]) and hence, for the case when $a \in F(f_a)$, our result is complementary to their result that infinity is an explosion point for the set of *fast endpoints*.

In order to obtain Theorem 5.1 we first show, in Section 5.3, the existence of continua in $A(f_a) \cup F(f_a)$ which separate every non-fast endpoint from infinity.

In Section 5.4, we show that the set of non-fast endpoints together with infinity is a totally separated set for Fatou's function. The techniques we use are similar to those used in Section 5.3. This is a stronger result than the one given in Section 5.2.

The existence of continua in $A(f) \cup F(f)$ for both exponential functions and Fatou's function has a connection with spiders' webs. In Section 5.5, we show that a subset of $A(f) \cup F(f)$ is a spider's web for both f_a , when $a \in F(f_a)$, and Fatou's function, giving a different way of proving the existence of spiders' webs which uses the blowing-up property of the Julia set.

5.2. Fatou's function

In this section we focus on Fatou's function $f(z) = z + 1 + e^{-z}$. Since $J(f)$ is a Cantor bouquet for Fatou's function we can consider the set of endpoints of $J(f)$ which we denote by $E(f)$. Note that Theorem 1.21 implies that Mayer's result that infinity is an explosion point for the set of endpoints for $f_a(z) = e^z + a$, $a < -1$, holds for Fatou's function (see also the proof of Theorem 5.9).

We present a consequence of Theorem 4.1 which is to the best of our knowledge the first topological result about *non-escaping* endpoints of a Cantor bouquet Julia set of a transcendental entire function. In particular, we show that, for Fatou's function, the set of non-escaping endpoints, $\hat{E}(f) = E(f) \setminus I(f)$ together with infinity is a totally separated set. Barański, Fagella, Jarque and Karpińska have recently showed that $\hat{E}(f)$ has full harmonic measure with respect to the Baker domain ([7, Example 1.6]). Note that the set $\hat{E}(f)$ is the radial Julia set of f , $J_r(f)$, as defined in [57]. This set was introduced by Urbański [80] and McMullen [45] and there are many known results regarding its dimension (see also [81]).

THEOREM 5.2. *Let $f(z) = z + 1 + e^{-z}$. Then $\hat{E}(f) \cup \{\infty\}$ is totally separated.*

PROOF. Suppose $p, q \in \hat{E}(f) \cup \{\infty\}$, with $p \neq q$. If $p = \infty$ then, since by Theorem 4.1 $I(f)$ is a spider's web, there is a continuum in $I(f)$ which separates p from q in $\hat{E}(f) \cup \{\infty\}$. If $p, q \neq \infty$ then we use the fact that the set of all the endpoints of f is totally separated

([2, Theorem 1.7]). Hence there exists a closed, connected set Δ as in Lemma 1.17. If Δ is bounded then the proof is complete. If Δ is unbounded then take U to be the component of $\hat{\mathbb{C}} \setminus \Delta$ that contains p and take a continuum γ in $I(f)$ which separates p from infinity. It follows that we can find a bounded simply connected domain $G \subset U$ with $p \in G$ and $\infty \notin \partial G$ and such that $\partial G \subset \Delta \cup \gamma$. Hence there exists a closed and connected set, ∂G , which separates p from q in $\hat{E}(f) \cup \{\infty\}$. Hence the result follows. \square

REMARK 5.2.1. Barański [5] studied a subclass of disjoint-type functions. Using his results and the function $z \mapsto e^{-1}ze^{-z}$, of which Fatou's function is a lift, we can show that $E(f)$ is totally disconnected but $E(f) \cup \{\infty\}$ is connected and hence deduce that $\hat{E}(f) \cup \{\infty\}$ is totally disconnected. We used this method in [34] but here we prove Theorem 5.2 which is stronger.

5.3. The exponential family

In this section we consider the family of functions $f_a(z) = e^z + a$, $a \in F(f_a)$. Our aim is to show that not only do the non-escaping endpoints together with infinity form a totally separated set but actually this result holds for the non-fast endpoints together with infinity.

We now give two lemmas that we need.

LEMMA 5.3. *Let $f_a(z) = e^z + a$, $a \in \mathbb{C}$. If $\operatorname{Re}(z) \geq \log 2|a|$, then*

$$(5.3.1) \quad \frac{1}{2}e^{\operatorname{Re}(z)} \leq |f(z)| \leq 2e^{\operatorname{Re}(z)}.$$

PROOF. In order to prove the first inequality of (5.3.1) note that

$$\begin{aligned} |f(z)| &\geq |e^z| - |a| \\ &= e^{\operatorname{Re}(z)} - |a| \\ &\geq \frac{1}{2}e^{\operatorname{Re}(z)}, \end{aligned}$$

since $e^t/2 \geq |a|$, for $t \geq \log 2|a|$.

For the second inequality of (5.3.1) we have

$$\begin{aligned} |f(z)| &\leq |e^z| + |a| \\ &= e^{\operatorname{Re}(z)} + |a| \\ &\leq 2e^{\operatorname{Re}(z)}, \end{aligned}$$

since $e^t \geq |a|$, for $t \geq \log|a|$. □

LEMMA 5.4. *Let $f_a(z) = e^z + a$, $a \in F(f_a)$ and $g(r) = e^{r/2}$. Then there exists $R_0 > 0$ such that, for $r \geq R_0$,*

$$I(f_a, (g^n(r))) = \{z : |f_a^n(z)| \geq g^n(r), n \in \mathbb{N}\} \subset A_r(f_a).$$

PROOF. We have

$$(5.3.2) \quad M(r, f_a) = M(r) \leq e^r + |a| \leq 2e^r, \quad r > \log|a|,$$

and we set $\tilde{M}(r) = 2e^r$. Then

$$g(r^3) = \exp(r^3/2) \geq 8 \exp(3r) = (\tilde{M}(r))^3, \text{ for } r > \max\{\log|a|, 3\}.$$

Hence

$$g^2(r^3) \geq g((\tilde{M}(r))^3) \geq (\tilde{M}^2(r))^3, \text{ for } r > \max\{\log|a|, 3\},$$

and by induction we deduce that

$$(5.3.3) \quad g^n(r^3) \geq (\tilde{M}^n(r))^3 \geq (M^n(r))^3 \geq M^n(r),$$

for $r > \max\{\log|a|, 3\}$. Take $R_1 > \max\{\log|a|, 3\}$ and $R_0 = R_2^3$ with $R_2 > R_1$. So, by (5.3.3), if $|f_a^n(z)| \geq g^n(r)$ for some $r \geq R_0$ then

$$|f_a^n(z)| \geq g^n(r) \geq g^n(R_2^3) \geq M^n(R_2), \text{ for } n \in \mathbb{N},$$

and so $z \in A_r(f_a)$. □

Let $E(f_a)$ be the set of endpoints of the curves in $J(f_a)$. We denote by $\hat{E}_A(f_a)$ the set of non-fast endpoints, defined by $\hat{E}_A(f_a) = E(f_a) \setminus A(f_a)$. We prove the following which is the main tool in the proof of Theorem 5.1.

THEOREM 5.5. *Let $f_a(z) = e^z + a$, where $a \in F(f_a)$ and let $R > 0$. Then every point in $\hat{E}_A(f_a)$ can be separated from infinity by a continuum $\gamma \subset A_R(f_a) \cup F(f_a)$.*

REMARK 5.3.1. Since, by [76] (see also [62] for a full description in a general setting), $A(f_a)$ consists of all the curves in $J(f_a)$ except for some of their endpoints we can see that $\mathbb{C} \setminus (A(f_a) \cup F(f_a)) = \hat{E}_A(f_a) \subset E(f_a)$.

PROOF. We will first construct a subset of $A_R(f_a) \cup F(f_a)$ and show that all its complementary components are bounded and then deduce that there exists a continuum $\gamma \subset A_R(f_a) \cup F(f_a)$ such that Theorem 5.5 holds. Note that $M(r, f_a) \geq r$ for all $r > 0$ and hence $A_R(f_a)$ is defined for all $R > 0$.

Since f_a does not have wandering domains [11, Theorem 12] and on any Siegel disc f_a must be univalent, the point a (the only singular value of f_a) must lie in an attracting or parabolic cycle of Fatou components [11, Theorem 7]. Let $U_0 \mapsto U_1 \mapsto \dots \mapsto U_p = U_0$ be the cycle of Fatou components of period p , labeled such that $a \in U_1$. Suppose first that $p \geq 2$. Then U_0 contains an entire left half-plane $\mathcal{L} = \{z : \operatorname{Re}(z) \leq -c\}$, where $c > 0$ will be chosen to be large enough shortly. Consider a horizontal curve σ in U_0 along which $\operatorname{Re}(z) \rightarrow -\infty$. Any preimage component of σ under f_a^{p-1} is a simple curve to ∞ , in particular, there is one, say σ' , in U_1 . If we join σ' to a with a curve lying entirely in U_1 we have a curve that joins a to ∞ and lies in U_1 . Its preimage under f_a consists of countably many curves joining $-\infty$ to $+\infty$ that lie in U_0 and are $2\pi i$ translates of each other. For a discussion on this see also [56, Chapter 4] and [75]. Now we consider an enumeration of these curves and denote by σ_m , $m \in \mathbb{N}$, the intersection of the unbounded component of the m -th such curve with $\{z : \operatorname{Re}(z) > -c\}$. We choose $c > 0$ such that $c > C = \max\{|\operatorname{Im}(z) - \operatorname{Im}(w)| : z \in \sigma_m, w \in \sigma_{m+1}\}$. Note that $2\pi \leq C < \infty$ since the preimages of U_0 contain horizontal half-lines of the form $\{x + yi : x > \alpha, y = (2j + 1)\pi\}$, for some $\alpha > 0$ and all $j \in \mathbb{Z}$. If $p = 1$ the argument is similar with $a \in U_0$.

Take

$$\mathcal{M} = \mathcal{L} \cup \bigcup_m \sigma_m \subset U_0,$$

and consider the sets

$$V_n = \mathcal{M} \cup f_a(\mathcal{M}) \cup \dots \cup f_a^n(\mathcal{M}), \quad n \in \mathbb{N}.$$

Let $R \geq R_1 = \max\{c, R_0, \log 2|a|\}$, where R_0 is the constant in Lemma 5.4, and let $g(R) = e^{R/2}$. We consider the set X defined by

$$X = \{z \in \mathbb{C} : \forall n \in \mathbb{N}, \text{ either } |f_a^n(z)| \geq g^n(R) \text{ or } f_a^n(z) \in \overline{V_n}\}.$$

Note that X is closed and also, by Lemma 5.4, and since $\overline{V_n} \subset F(f_a)$, we have that $X \subset A_R(f_a) \cup F(f_a)$. We will show that all the components of X^c are bounded.

To show that all the components of X^c are bounded we first define

$$B_n = \{x + yi : -c < x < g^n(R), -g^n(R) < y < g^n(R)\}, \quad \text{for } n \in \mathbb{N},$$

and

$$B = \{z \in \mathbb{C} : \forall n \in \mathbb{N}, |f_a^n(z)| \notin B_n \text{ or } f_a^n(z) \in \overline{V_n}\}.$$

Note that $B \subset X$ and also that $\mathcal{M} \subset B$ since $f_a^n(\mathcal{M}) \subset V_n, n \in \mathbb{N}$.

We will show that all the components of B^c are bounded. As $B \subset X$, this will imply that all the components of X^c are bounded. Suppose that there exists an unbounded component of B^c , say V . We will construct a certain bounded curve $\gamma_0 \subset V$ and by considering the images $f^k(\gamma_0), k \geq 0$, we will deduce, by Lemma 4.8, that γ_0 contains a point of B , giving a contradiction.

Let $S_{0,m}, m \geq 0$, be the region contained between σ_m and σ_{m+1} which lies in the half-plane $\{z : \operatorname{Re}(z) > -c\}$. As B is closed, V is open and hence there exists $m_0 \geq 0$ such that V contains $z_{0,1}, z_{0,2} \in S_{0,m_0}$ and a curve $\gamma_0 \subset V$ joining $z_{0,1}$ to $z_{0,2}$ such that:

- (a) $\operatorname{Re}(z_{0,1}) = a_0 > R$
- (b) $\operatorname{Re}(z_{0,2}) = a_0 + 10$
- (c) $\gamma_0 \subset \{z : a_0 \leq \operatorname{Re}(z) \leq a_0 + 10\} \cap S_{0,m_0}$
- (d) $\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_0$,

where $S_{0,m_0}, m_0 \geq 0$, is a region defined as above.

Now, for $k \geq 0$ and $m \in \mathbb{N}$, consider $S_{k+1,m}$ to be the intersection of B_{k+1}^c with the region contained between σ_m and σ_{m+1} which lies in the half-plane $\{z : \operatorname{Re}(z) > -c\}$.

Since $\mathcal{M} \subset B$, γ_0 lies in some $S_{0,m_0}, m_0 \geq 0$.

We will use induction to show that there exists a family of curves γ_k , $k \geq 0$, with $\gamma_{k+1} \subset f_a(\gamma_k)$, such that, for all $k \geq 0$, γ_k lies in some S_{k,m_k} and joins the points $z_{k,1}, z_{k,2}$ such that:

- (a) $\operatorname{Re}(z_{k,1}) = a_k > g^k(R)$
- (b) $\operatorname{Re}(z_{k,2}) = a_k + 10$
- (c) $\gamma_k \subset \{z : a_k \leq \operatorname{Re}(z) \leq a_k + 10\} \cap S_{k,m_k}$
- (d) $\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_k$.

By assumption this is true for $k = 0$. Suppose that it holds for $0 \leq j \leq k$; we show that it is also true for $j = k + 1$.

It follows from (a), (c) and Lemma 5.3 that

$$(5.3.4) \quad |f_a(z)| \geq \frac{e^{a_k}}{2} > \frac{e^{g^k(R)}}{2} \geq \sqrt{2}g^{k+1}(R), \text{ for } z \in \gamma_k,$$

as $e^t > 2\sqrt{2}e^{t/2}$, for $t \geq 3$. Hence, the curve $f_a(\gamma_k)$ lies outside B_{k+1} and because it must not intersect with the curves σ_m , it lies in some region $S_{k+1,m_{k+1}}$. Indeed if $f_a(\gamma_k)$ meets B_{k+1} then there exists a point in γ_0 which belongs to B and hence we obtain a contradiction. We also have, by Lemma 5.3,

$$(5.3.5) \quad |f_a(z_{k,2})| \geq \frac{e^{\operatorname{Re}(z_{k,2})}}{2} = \frac{e^{10}}{2}e^{a_k} \geq \frac{e^{10}}{4}|f_a(z_{k,1})|.$$

Since $S_{k+1,j_{k+1}} \subset \{z : \operatorname{Re}(z) \geq -c\}$, it follows from (5.3.4) and (5.3.5) by a routine calculation that the set

$$\left\{ w \in f_a(\gamma_k) : |w| \geq \frac{e^{10}}{8}|f_a(z_{k,1})| \right\}$$

contains points $z_{k+1,1}, z_{k+1,2}$ and a curve $\gamma_{k+1} \subset f_a(\gamma_k)$ joining them such that

- (a) $\operatorname{Re}(z_{k+1,1}) = a_{k+1} > g^{k+1}(R)$
- (b) $\operatorname{Re}(z_{k+1,2}) = a_{k+1} + 10$
- (c) $\gamma_{k+1} \subset \{z : a_{k+1} \leq \operatorname{Re}(z) \leq a_{k+1} + 10\} \cap S_{k+1,m_{k+1}}$
- (d) $\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_{k+1}$,

as required.

As $\gamma_{k+1} \subset f_a(\gamma_k)$, for $k \geq 0$, it follows (see, Lemma 4.8) that there exists $\zeta \in \gamma_0$ such that $f_a^k(\zeta) \in \gamma_k$, for all $k \geq 0$. Hence,

$$|f_a^k(\zeta)| \geq \sqrt{2}g^k(R), \text{ for } k \geq 0,$$

and so $\zeta \in B$, which gives us a contradiction. Therefore, all the components of B^c and so all the components of X^c are bounded.

Now take a point $z \in \hat{E}_A(f_a)$. Then z belongs to a component of X^c , say V' . Since V' is bounded there exists a continuum $\gamma \subset \partial V'$ that separates z from ∞ . As X is closed we have found a continuum in X which separates z from ∞ . This completes the proof if $R \geq R_1$. If $R < R_1$ then $A_{R_1}(f_a) \cup F(f_a) \subset A_R(f_a) \cup F(f_a)$ and hence $\gamma \subset A_R(f_a) \cup F(f_a)$. \square

We can now give the main result of this chapter.

THEOREM 5.6. *Let $f_a(z) = e^z + a$, where $a \in F(f_a)$. Then $\hat{E}_A(f_a) \cup \{\infty\}$ is totally separated.*

PROOF. Suppose $p, q \in \hat{E}_A(f_a) \cup \{\infty\}$, with $p \neq q$. If $p = \infty$ then, by Theorem 5.5, there is a continuum in $A(f_a) \cup F(f_a) \subset (\hat{E}_A(f_a) \cup \{\infty\})^c$ which separates p from q . If $p, q \neq \infty$ then we use the fact that the set of all endpoints of f_a is totally separated ([2, Theorem 1.7]). Hence there exists a closed, connected set Δ as in Lemma 1.17. If Δ is bounded then the proof is complete. If Δ is unbounded then take U to be the component of $\hat{\mathbb{C}} \setminus \Delta$ that contains p and take a continuum γ in $A(f_a) \cup F(f_a)$ which separates p from infinity. It follows that we can find a bounded simply connected domain $G \subset U$ with $p \in G$ and $\infty \notin \partial G$ and such that $\partial G \subset \Delta \cup \gamma$. Hence there exists a closed and connected set, ∂G , which separates p from q in $\hat{E}_A(f_a) \cup \{\infty\}$. Hence the result follows. \square

5.4. Fatou's function; a stronger result

In Section 5.2 we showed that, for Fatou's function, the set of non-escaping endpoints together with infinity is a totally separated set. Here we show that similar methods to those used in Section 5.3 can be used to prove that this result is also true for the non-fast endpoints together with infinity. Following the structure of Section 5.3, we first show that every non-fast endpoint can be separated from infinity by a continuum in $A_R(f) \cup F(f)$, for some $R > 0$.

In order to do this we need a lemma similar to Lemma 5.4 which we state without proof. We have the following result.

LEMMA 5.7. *Let $f(z) = z + 1 + e^{-z}$ and $g(r) = e^{r/2}$. Then, there exists $R'_0 > 0$ such that, for $r \geq R'_0$,*

$$I(f, (g^n(r))) = \{z : |f^n(z)| \geq g^n(r), n \in \mathbb{N}\} \subset A_{R'_0}(f).$$

Now we outline the proof of the following result. The proof is similar to the proof of Theorem 5.5.

THEOREM 5.8. *Let $f(z) = z + 1 + e^{-z}$ and let $R' > 0$. Then every point in $\hat{E}_A(f)$ can be separated from infinity by a continuum $\gamma \in A_{R'}(f) \cup F(f)$.*

PROOF. Note first that $M(r, f) \geq r$ for all $r > 0$ and hence $A_R(f)$ is defined for all $R > 0$.

Let $R' \geq R'_1 = \max\{3, R'_0\}$, where R'_0 is the constant in Lemma 5.7, and let $g(r) = e^{r/2}$. First, we consider the half-plane $\mathcal{R} = \{z : \operatorname{Re}(z) \geq 1\} \subset F(f)$ and the half-lines $\sigma_j = \{x + yi : x < 1, y = 2j\pi\}, j \in \mathbb{Z}$. We define the set

$$C = \mathcal{R} \cup \bigcup_{j \in \mathbb{Z}} \sigma_j.$$

Since σ_j is mapped into \mathcal{R} under f , for $j \in \mathbb{Z}$, and $f^n(C) \subset \mathcal{R} \subset F(f)$, for $n \in \mathbb{N}$, we deduce that $C \subset F(f)$ and C is forward invariant under f .

Now we consider the set

$$X = \{z \in \mathbb{C} : \forall n \in \mathbb{N}, \text{ either } |f^n(z)| \geq g^n(R') \text{ or } f^n(z) \in C\}.$$

Note that X is closed and also, by Lemma 5.7, and since $C \subset F(f)$, we have that $X \subset A_{R'}(f) \cup F(f)$. We will show that all the components of X^c are bounded.

We define

$$B_n = \{x + yi : -g^n(R') < x < 1, -g^n(R') < y < g^n(R')\},$$

for $n \in \mathbb{N}$,

and

$$B = \{z : f^n(z) \notin B_n, \text{ for all } n \in \mathbb{N}\}.$$

Then $B \subset X$ and B is closed.

We will show that all the components of B^c are bounded. As $B \subset X$, this will imply that all the components of X^c are bounded.

Now, for $k \geq 0$, consider the half-strips

$$S_{k+1,j} = \{x + yi : x < 1, 2(j-1)\pi < y < 2j\pi\} \cap B_{k+1}^c,$$

where $j \in \mathbb{Z}$.

Suppose that there exists an unbounded component U of B^c . As in the proof of Theorem 5.5, we construct a family of curves γ_k , $k \geq 0$, with $\gamma_{k+1} \subset f(\gamma_k)$ and $\gamma_0 \subset U$ and by considering the images $f^k(\gamma_0)$, $k \geq 0$, we shall deduce, by [72, Lemma 1], that γ_0 contains a point of B , which will give a contradiction. To be precise, we shall deduce that, for all $k \geq 0$, we have a curve γ_k which lies in some half-strip S_{k,j_k} , $j_k \in \mathbb{Z}$, and joins the points $z_{k,1}, z_{k,2}$ such that:

- (a) $\operatorname{Re}(z_{k,1}) = a_k < -g^k(R')$
- (b) $\operatorname{Re}(z_{k,2}) = a_k - 10$
- (c) $\gamma_k \subset \{z : a_k - 10 \leq \operatorname{Re}(z) \leq a_k\} \cap S_{k,j_k}$
- (d) $-\operatorname{Re}(z) \geq |z|/2$, for $z \in \gamma_k$.

Suppose that (a)-(d) hold for some $k \geq 0$. Since U is unbounded we can choose, for $k = 0$, a curve γ_0 satisfying (a)-(d). Then $f(\gamma_k)$ lies outside B_{k+1} and because it must avoid the half-lines $\sigma_j = \{x + yi : x < 1, y = 2j\pi\}$, it lies in some half-strip $S_{k+1,j_{k+1}}$. Hence, by using Lemma 4.7, we can construct a curve $\gamma_{k+1} \subset f(\gamma_k)$ satisfying (a)-(d) with k replaced by $k + 1$.

As $\gamma_{k+1} \subset f(\gamma_k)$, for $k \geq 0$, it follows from Lemma 4.8 that there exists $\zeta \in \gamma_0$ such that $f^k(\zeta) \in \gamma_k$, for all $k \geq 0$. Hence,

$$|f^k(\zeta)| > \sqrt{2}g^k(R'), \text{ for } k \geq 0,$$

and so $\zeta \in B$, which gives us a contradiction.

Therefore, since $B^c \neq \emptyset$, we have shown that all the components of B^c and so all the components of X^c are bounded.

Similarly to the proof of Theorem 5.5 we can now take a point $z \in \hat{E}_A(f)$ which belongs to a component of X^c and find a continuum in X which separates z from ∞ . This

completes the proof if $R' \geq R'_1$. If $R' < R'_1$ then $A_{R'_1}(f) \cup F(f) \subset A_R(f) \cup F(f)$ and hence $\gamma \subset A_R(f) \cup F(f)$ \square

As in the proof of Theorem 5.6, we can now deduce the following result which implies Theorem 5.2, as noted earlier.

THEOREM 5.9. *Let $f(z) = z + 1 + e^{-z}$. Then $\hat{E}_A(f) \cup \{\infty\}$ is totally separated.*

PROOF. Suppose that $p, q \in \hat{E}_A(f) \cup \{\infty\}$, with $p \neq q$. If $p = \infty$ then, by Theorem 5.8, there is a continuum in $A(f) \cup F(f)$ which separates p from q .

Note now that $J(f)$ is a Cantor bouquet and hence by Theorem 1.21 we deduce that the set of *all* endpoints of f is totally separated. Hence if $p, q \neq \infty$ we can proceed as in the proof of Theorem 5.6. \square

5.5. Spiders' webs

In the last section of this chapter we prove that there exist $R, R' > 0$ such that $A_R(f_a) \cup F(f_a)$, for $a \in F(f_a)$, and $A_{R'}(f) \cup F(f)$, where f is Fatou's function, are spiders' webs. In order to do this, we will make use of Theorems 5.5 and 5.8 respectively. Unlike in Section 5.2, here we deduce the results on spiders' webs by the results that we prove for the non-escaping endpoints in the previous two sections.

We have the following result.

THEOREM 5.10. *Let $f_a(z) = e^z + a$, where $a \in F(f_a)$. Then, for all $R > 0$, $A_R(f_a) \cup F(f_a)$ is a spider's web.*

PROOF. In Theorem 5.5 we constructed a set $X \subset A_R(f_a) \cup F(f_a)$, and we showed that all the components of X^c are bounded. Take a bounded component of X^c , say G , which contains a non-escaping endpoint $w \in J(f_a)$. As X is a closed set, we deduce that G is open and $\partial G \subset X \subset A_R(f_a) \cup F(f_a)$.

Now let $G_n = \widetilde{f_a^n(G)}$, where \widetilde{U} denotes the union of U and its bounded complementary components. We show that there exist domains G_n and a sequence (n_k) such that

$$(5.5.1) \quad G_{n_k} \subset G_{n_{k+1}} \text{ and } \partial G_{n_k} \subset A_R(f_a) \cup F(f_a), \text{ for } k \in \mathbb{N}, \text{ and } \bigcup_k G_{n_k} = \mathbb{C}.$$

First note that, for each $n \in \mathbb{N}$, $\partial G_n \subset f_a^n(\partial G)$, since G is a bounded domain. Also, as $\partial G \subset X \subset A_R(f_a) \cup F(f_a)$, and both $A_R(f)$ (see [68]) and $F(f_a)$ are forward invariant under f_a we have that

$$(5.5.2) \quad \partial G_n \subset f_a^n(\partial G) \subset A_R(f_a) \cup F(f_a).$$

Now consider the disc $D(0, g^n(R))$, $n \in \mathbb{N}$. Note that $D(0, g^n(R))$ may contain a , which is the only exceptional point for f_a . Since G contains a neighbourhood of w , the blowing-up property (Lemma 1.1) implies that there exists $M = M(n) \in \mathbb{N}$ such that $G_M \supset D(0, g^n(R))$. More precisely, $f_a^M(G)$ covers $D(0, g^n(R))$ except for a neighbourhood of a and so $G_M = \widetilde{f_a^M(G)} \supset D(0, g^n(R))$. Therefore, ∂G_M surrounds $D(0, g^n(R))$. This argument together with (5.5.2) imply that there exists a sequence (n_k) such that (5.5.1) is true.

Recall that for every transcendental entire function every component of $A_R(f)$ is unbounded. In particular, in our case, $A_R(f_a)$ consists of unbounded curves in the pinched Cantor bouquet. Moreover, every component of $F(f_a)$ contains a curve to $+\infty$, as we mentioned in the proof of Theorem 5.5, and so it is unbounded. All the components of $A_R(f_a)$ and $F(f_a)$ will meet all ∂G_{n_k} , for k large enough. Hence $A_R(f_a) \cup F(f_a)$ contains a spider's web and it is connected. Indeed, take two points $x, y \in A_R(f_a) \cup F(f_a)$ which belong to different unbounded components of either $A_R(f_a)$ or $F(f_a)$. Since both components meet all ∂G_{n_k} , for k large enough, we can join x to y by a continuum lying entirely in $A_R(f_a) \cup F(f_a)$. Therefore, we deduce that $A_R(f_a) \cup F(f_a)$ is a spider's web. \square

An argument similar to the one used in the above proof can be adapted to show that, for Fatou's function, we have a result analogous to Theorem 5.10. Hence we only outline the proof of the following theorem.

THEOREM 5.11. *Let $f(z) = z + 1 + e^{-z}$. Then, for all $R' > 0$, $A_{R'}(f) \cup F(f)$ is a spider's web.*

PROOF. Let $R' > 0$ and X be as in Theorem 5.8. Then Theorem 5.8 implies that we can find a bounded component of X^c that contains a non-escaping endpoint. We proceed in the same way as in Theorem 5.10 and since $F(f)$ consists of one unbounded component, a Baker domain, we can show that $A_{R'}(f) \cup F(f)$ is a spider's web. \square

CHAPTER 6

Plans for future work

There are several interesting questions that follow naturally from the recent results on spiders' webs and endpoints that were described in Chapters 4 and 5. The aim of this chapter is to present some of these questions that will be investigated in future work.

In particular, we aim to identify the greatest possible class of functions with Baker domains for which $I(f)$ is a spider's web, which will result in a generalisation of existing results. In addition, we will look at the case where $I(f)$ is not a spider's web for such functions. Finally, it seems natural to ask if we can obtain results on non-escaping endpoints for larger classes of functions. More specifically, our priority will be to look at functions studied in [8] for which the set of endpoints is totally separated. Hence, the questions that arise fall into the following three categories.

Question 1 What are the different classes of functions with Baker domains for which $I(f)$ is a spider's web?

There are many interesting results for functions with different number and types of Baker domains (see, for example, [65], [7]). In these papers there are concrete examples of functions for which we have a lot of information regarding their dynamical behaviour. The idea is to work on generalising recent results on spiders' webs to classes of functions with more than one Baker domain (even infinitely many) by developing the techniques introduced in this thesis in order to show that $I(f)$ or $A(f) \cup F(f)$ is a spider's web.

More precisely, in Chapter 4 we developed a completely new technique to show that Fatou's function has the property that $I(f)$ is a spider's web and deduced that the same property holds for a family of functions, all of which have exactly one Baker domain. We also gave a result (Theorem 4.14) which describes a large class of functions for which

$A(f) \cup F(f)$ is a spider's web. Moreover, in Theorem 5.10 we presented another technique for showing that a set is a spider's web which is based on first constructing continua in the set and then using the blowing-up property of $J(f)$. A very important tool in all of the above cases is the use of suitably chosen subsets of $I(f)$ with a uniform rate of escape.

Using one of these three techniques, adapting them or even developing new ones we aim to find new families of functions for which $I(f)$ is a spider's web.

Question 2 Are there examples of functions with Baker domain(s) for which $I(f)$ is not a spider's web? What can we say in this case?

While working on functions with Baker domain(s) and trying to describe their escaping sets we may find some functions for which $I(f)$ is not a spider's web. It will be interesting to investigate what is true in this case. Our priority will be to check whether we can apply Theorem 4.14 or adapt Theorem 5.10 to show that $I(f) \cup F(f)$ or $A(f) \cup F(f)$ is a spider's web. Moreover, we can examine whether $I(f)$ has the structure of a *weak spider's web*.

In cases when a set is not a spider's web, or cannot be proved to be one, the need for a different structure arises. Osborne, Rippon and Stallard have recently introduced [53] the term *weak spider's web* to describe a connected set that has no unbounded closed connected sets in its complement. A weak spider's web can still give us information about the structure of a set. As this is a very new idea little is known about it.

Hence starting with specific examples of functions we aim to come up with classes of functions for which we can show that $I(f)$ is not a spider's web. Then we will investigate whether we can still extract some useful information about the structure of $I(f)$ by looking at supersets of it or/and other possible structures that $I(f)$ may have.

Question 3 Are there classes of functions outside the exponential family for which the set of non-escaping endpoints together with infinity form a totally separated set?

In [2] Alhabib and Rempe-Gillen considered functions that are not in the exponential family and looked at their escaping endpoints. In particular, they proved the following result ([2, Theorem 1.9]). Here $\tilde{E}(f)$ denotes the set of escaping endpoints.

THEOREM 6.1. (Alhabib and Rempe-Gillen, 2015) *Let f be a finite-order entire function in the class \mathcal{B} , a finite composition of such functions, or, more generally, a function satisfying a ‘uniform head-start condition’ in the sense of [74]. Then the set $\tilde{E}(f) \cup \{\infty\}$ is connected.*

If, additionally, f is hyperbolic, then ∞ is an explosion point for $\tilde{E}(f)$.

Hence it is natural to ask if our result on the non-escaping endpoints can be extended to other classes of functions.

An important ingredient in our proofs of the results on the non-escaping endpoints is the fact that the set of all endpoints is totally separated for the functions considered. Hence for us the most natural family of functions to consider is the one considered in [8] for which $J(f)$ is a Cantor bouquet (see Theorem 1.13). Recall that this family consists of disjoint-type functions of finite order and finite compositions of such functions. For this class of functions Theorem 1.21 implies that the set of all endpoints is totally separated.

The next step is to prove the existence of continua that separate the non-escaping endpoints from infinity. In order to prove our results in Chapter 5 we used two slightly different techniques: the first one was based on showing that a suitable set has the structure of a spider’s web and the second one was based on a direct construction of continua which lie in the complement of $\hat{E}_A(f)$ and separate every non-fast escaping endpoint from infinity. For Question 3 to be answered, new techniques may need to be developed, as well as considering the techniques mentioned above, in order to show that a suitable set has some structure that contains these continua. General results on spiders’ webs obtained during the work on Question 1 may be used as well. This may lead to much further generalisations of the recent results on non-escaping endpoints.

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